The Sine Rule Disguised

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INTRODUCTION

Mathematics education is more than simply chasing after good marks in the matric examination. Of course, good marks are important, but good mathematics education should also develop what Skemp (1974) has called 'relational understanding'. Most examinations still focus largely on testing 'instrumental proficiency' – i.e. the ability to apply and carry out learnt procedures and algorithms correctly. In contrast, relational understanding is about developing meaningful connections between concepts, as well as developing the ability to think a bit more creatively, to be able to solve novel problems, and to begin posing one's own problems. Of course, relational understanding is much harder to establish and assess, but that doesn't mean we shouldn't constantly strive towards this as a goal.

A particularly useful problem-posing strategy that one can readily nurture in one's students is the so-called "what if" strategy (Brown & Walter, 1990). For example, after encountering a particular result about triangles in class, it would be natural to consider what would happen if the triangle was a quadrilateral, or a pentagon. While not all such questions necessarily lead to meaningful further exploration, some may, so it is important to continue cultivating such a mindset in students, as discovering something for oneself as a learner is immensely rewarding and empowering.

WHAT IF?

Here is a simple example to consider. Take any triangle. What would happen if we now construct arbitrary triangles on each side of the original triangle as illustrated in Figure 1? Are there any features that remain invariant, i.e. unchanged? Readers may wish to explore such a context by replicating the situation depicted in Figure 1 with their own dynamic geometry software. Alternatively, an online dynamic sketch is available for readers to explore interactively at:

http://dynamicmathematicslearning.com/invariant-product-triangles-on-sides-plus-Anghel.html

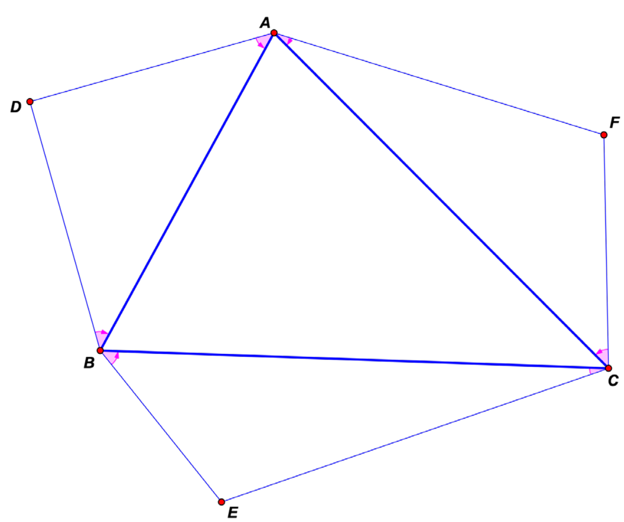

FIGURE 1: Arbitrary triangles on the sides of a central triangle

At first glance this may seem like a silly, rather meaningless scenario to consider. After all, what could possibly remain invariant (unchanged) in such a situation? However, the configuration has more than one property that remains invariant. The most obvious one is if we consider the (convex or concave) hexagon $ADBECF$ which has an angle sum of 360° (which is easy to see, and prove, from the diagram). Perhaps not so easy to 'see', however, is the following hidden invariant result involving the product of the ratios of the sides and the sine ratios of the angles of the respective outer triangles:

$$\frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA} \times \frac{\sin D\hat{A}B}{\sin D\hat{B}A} \times \frac{\sin E\hat{B}C}{\sin E\hat{C}B} \times \frac{\sin F\hat{C}A}{\sin F\hat{A}C} = 1$$

Note that the result remains true even if the hexagon $ADBECF$ becomes concave or crossed, and readers are encouraged to confirm this for themselves by dragging points D, E and F in their own sketches or using the interactive sketch at the link given earlier.

PROOF

Why is this somewhat surprising relationship true, and how can we prove it? The result may remind one of the sine rule, and it is indeed simply the sine rule disguised in a somewhat novel situation. Consider, for example ΔADB. Applying the sine rule to it we obtain:

$$\frac{AD}{\sin D\hat{B}A} = \frac{DB}{\sin D\hat{A}B} \quad \Rightarrow \quad \frac{AD}{\sin D\hat{B}A} \times \frac{\sin D\hat{A}B}{DB} = 1 \quad \Rightarrow \quad \frac{AD}{DB} \times \frac{\sin D\hat{A}B}{\sin D\hat{B}A} = 1$$

Since the respective products of the other ratios in triangles BEC and CFA are similarly equal to 1, the result follows.

GENERALISATION

It is now also easy to see that the result will similarly generalise to any polygon with triangles constructed on the sides, and is yet another illustration of the so-called 'discovery' function of proof.

REFERENCES

Brown, S.I. & Walter, M. I. (1990). *The art of problem posing*. London: Routledge.

Skemp, R. (1974). Relational understanding and instrumental understanding. *Mathematics Teaching*, *77*, 20-26.