Investigating Alternate Angle Sums of Crossed Cyclic Hexagons

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INTRODUCTION

Euclid's Proposition 22 in Book III (Joyce, 1996) famously states that "The sum of the opposite angles of quadrilaterals in circles equals two right angles." Since 'opposite' angles in a quadrilateral can also be viewed as 'alternate' angles, this theorem for (convex) cyclic quadrilaterals nicely generalizes to the following hexagon theorem[1]:

The sum of any set of alternate angles of a (convex) cyclic hexagon equals 360°

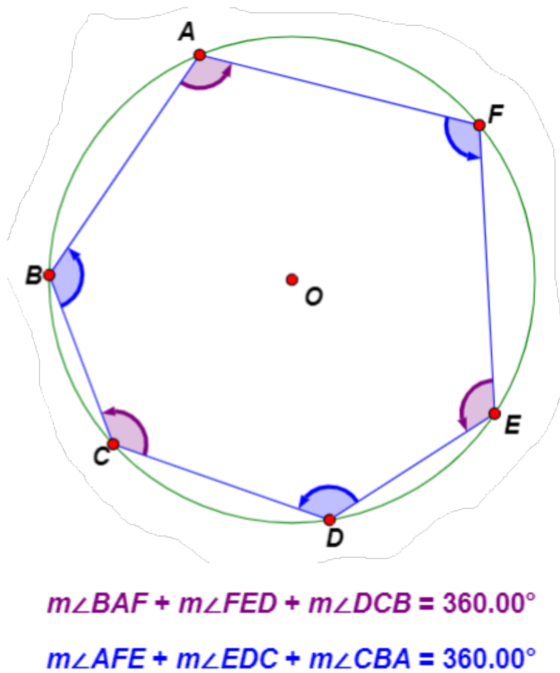

FIGURE 1

This generalization to (convex) cyclic hexagons has been used in a learning activity with dynamic geometry in De Villiers (1999) with both high school learners and prospective mathematics teachers. It was also used as an examination question in a Grade 11 paper by Duncan Samson (2017) in which learners at their school produced multiple creative solutions. More recently, Yiu-Kwong Man (2023) provided a neat visual proof of the convex case of this cyclic hexagon theorem. Since the *converse* of this (convex) cyclic hexagon theorem is false, unlike the case for cyclic quadrilaterals, it also provides a valuable educational opportunity for learners to explore further (De Villiers, 2006).

[1] The earliest mention and proof of this result seems to be that of Duncan Gregory (1836). This result has also been called Turnbull's theorem by MacKinnon (1990) after a schoolboy who rediscovered it in class.

WHAT IF?

In a dynamic geometry environment, it seems natural to ask what would happen to the sums of the alternate angles if some of the vertices were dragged past some of the others on the circumference of the circle, and the cyclic hexagon becomes crossed as shown in Figure 2. Such a strange, surprising finding immediately raises the intriguing question of why the two sums of alternate angles are now equal to 180°. Readers are invited to dynamically explore this particular configuration using directed angles[2], as well as other possible crossed configurations, with a *Sketchpad* sketch at http://dynamicmathematicslearning.com/cyclichex.html (click on the 'Link to cyclic crossed hexagon - directed angles' button within the sketch to navigate to it).

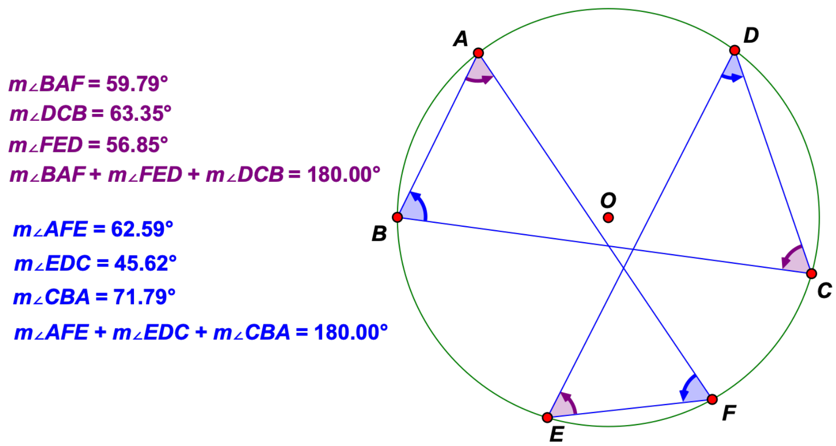

FIGURE 2

At the preceding URL, the reader should find that the sum of alternate angles for a crossed cyclic hexagon is either $-180°$, 0° or 180°, depending on the configuration. To explain why (or prove that) these results are true is not hard and is well within range of average high school learners as shown below in 'proofs without words' for each of the three different configurations.

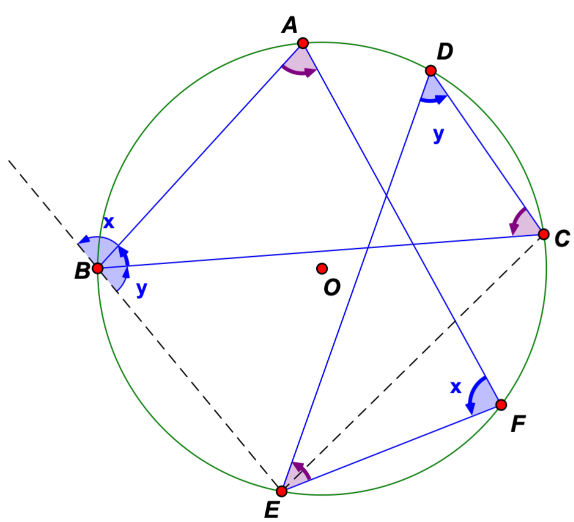

FIGURE 3: Alternate angle sum equals 180° (all angles positive)

[2] In this dynamic sketch, anti-clockwise angles are considered as positive whereas clockwise angles are regarded as negative. This corresponds to the standard convention in trigonometry.

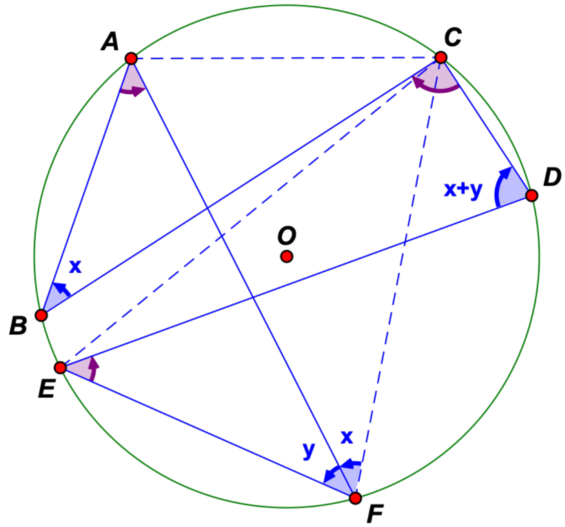

FIGURE 4: Alternate angle sum equals 0° (sum of two positive angles equals negative angle, or vice versa)

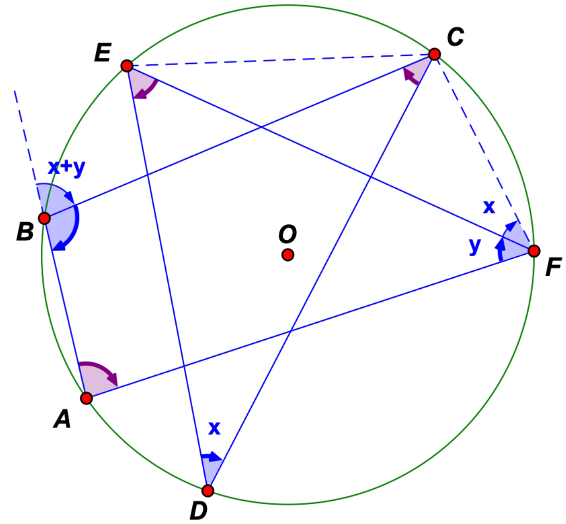

FIGURE 5: Alternate angle sum equals −180° (all angles negative)

It should be noted that there is considerable variation in how the vertices of the hexagon can be positioned in each of these three configurations. Consider, for example, the configuration shown in Figure 3. The alternate angle sum would not change if F remained on the arc $ADCE$ (since F remains subtended by the same chord AE and on the same side of the chord). Hence, the two configurations shown in Figure 6 are equivalent to the one shown in Figure 3 with all three angles at vertices B, D and F remaining positive. However, notice that in regard to the second diagram in Figure 6, that while the angles at vertices A and C remain positive, the angle at E has changed to negative. But it is easy to see that the sum of the angles at alternate vertices A, C and E remains equal to 180°, and is left to the reader to verify.

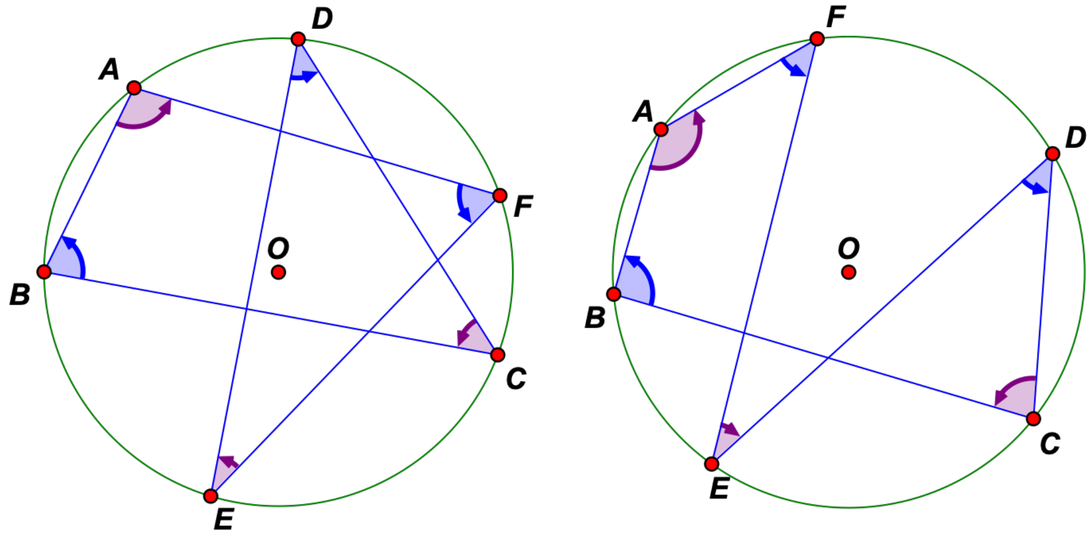

FIGURE 6: Equivalent configurations to Figure 3

Since the same principle of moving vertices along the same arc applies to any of the other vertices in Figures 3 to 5, we can obtain many different variations of the same three basic configurations.

Also note that if directed angles are used in the same way for a convex hexagon that the sum of the alternate angles could be $-360°$ if all its angles are negative (in other words, the vertices lie in a clockwise direction around the circumference of the circle).

CONCLUDING COMMENTS

The variety of possible configurations one can obtain for crossed cyclic hexagons provides a novel context and useful exercise for learners to apply their knowledge of circle geometry. The investigation can also easily be extended to cyclic octagons, and to cyclic $2n$-gons in general as shown in De Villiers (1993). With the aid of dynamic geometry, it provides a surprising but accessible challenge and a fruitful educational experience for average as well as talented learners.

REFERENCES

De Villiers, M. (1993). A unifying generalization of Turnbull's theorem. *International Journal of Mathematical Education in Science & Technology*, Vol. 24, No. 2, pp. 191-196.

De Villiers, M. (First published 1999). *Rethinking Proof with Sketchpad*. Emeryville: Key Curriculum Press.

De Villiers, M. (2006). Recycling cyclic polygons dynamically. *Mathematics in School*, Vol. 35, No. 3, pp. 2-4.

Gregory, D. (1836). Geometrical theorem. *Cambridge Mathematical Journal*, Vol. 1, p. 192.

Joyce, D. E. (1996). *Euclid's Elements* at: http://aleph0.clarku.edu/~djoyce/elements/bookIII/propIII22.html (Accessed on 3 July 2023).

MacKinnon, N. (1990). The Starship Enterprise and cyclic hexagons. *The Mathematical Gazette*, (March), Note 74.01, p. 40.

Man, Y-K. (2023). A visual proof of a theorem of cyclic hexagons. *Learning & Teaching Mathematics*, No. 34, p. 28.

Samson, D. (2017). Euclidean geometry – nurturing multiple solutions. *Learning & Teaching Mathematics*, No. 23, pp. 15-18.