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Isogonic Centres of a Triangle.

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THEOREM 1

*If on the sides of a triangle ABC equilateral triangles LBC MCA NAB be described externally, AL BM CN are equal and concurrent.**

FIGURE 28

For triangles BAM NAC are congruent,
since two sides and the contained angle in the one are equal to two sides and the contained angle in the other;

therefore BM = NC.

Similarly NC = AL.

Let BM CN meet at V. Join AV LV

Since $\angle VBA = \angle VNA$

therefore V lies on the circumcircle of ABN

and $\angle AVB = 120°$

Similarly V lies on the circumcircle of CAM

and $\angle CVA = 120°$

therefore $\angle BVC = 120°$

therefore V lies on the circumcircle of BCL

Hence $\angle LVC = \angle LBC = 60°$,

and AVL is a straight line.

THEOREM 2

If on the sides of a triangle ABC equilateral triangles L'BC M'CA N'AB be described internally, AL' BM' CN' are equal and concurrent

* T. S. Davies in the *Gentleman's Diary* for 1830, p. 36.

FIGURE 28

The demonstration of the previous case may be easily modified to suit this one.

Let the point of concurrency be denoted by V′.

The points V V′ are called the *isogonic centres* of ABC.

POSITIONS OF V AND V′

V

When each of the angles A B C is less than 120°, V is inside ABC.

When any one of the angles A B C is greater than 120°, V is outside ABC. For example, if A be greater than 120°, V lies between BA and CA produced.

When ABC is equilateral, its centroid, circumcentre, incentre, orthocentre all coincide, and V coincides with them.

V′

When B is greater than 60° and C is less than 60°, V′ is outside ABC and between CA and CB produced.

When B is less than 60° and C is greater than 60°, V′ is outside ABC and between BA and BC produced.

When B and C are both greater than 60°, or both less than 60°, V′ is outside ABC and between AB and AC produced.

When one of the angles of ABC, for example B, is 60°, V′ coincides with B.

When ABC is equilateral, V′ may be anywhere on the circumcircle of ABC.

Theorem 1 is closely connected with the problem

To find that point the sum of whose distances from the vertices of a triangle is a minimum.

This problem, Viviani relates, was proposed by Fermat to Torricelli, and by him handed over as an exercise to Viviani, who gives* the following construction for obtaining the point.

Let ABC be the triangle, and let each of its angles be less than 120°.

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* In the Appendix to his treatise *De Maximis et Minimis*, pp. 144, 150 (1659).

On AB and AC describe segments of circles containing angles of 120°. The arcs of these segments will intersect at the point required.

Viviani's proof that this is the point required is too long to extract.

Thomas Simpson in his *Doctrine and Application of Fluxions*, § 36 (1750) gives the following construction for determining the same point.

Describe on BC a segment of a circle to contain an angle of 120°, and let the whole circle BCQ be completed. From A to Q, the middle of the arc BQC, draw AQ intersecting the circumference of the circle in V, which will be the point required.

In § 431, Simpson treats the more general problem,

Three points A B C being given, to find the position of a fourth point P, so that if lines be drawn from thence to the three former, the sum

$$a \,.\, AP + b \,.\, BP + c \,.\, CP$$

where a b c denote given numbers, shall be a minimum.

Both the particular and the more general problem are discussed † by Nicolas Fuss in his memoir "De Minimis quibusdam geometricis, ope principii statici inventis" read to the Petersburg Academy of Sciences on 25th February 1796. In this memoir, denoting AV + BV + CV by s, Fuss gives the expressions

$$AV = \tfrac{1}{3}s + \frac{b^2 + c^2 - 2a^2}{3s}$$

$$BV = \tfrac{1}{3}s + \frac{c^2 + a^2 - 2b^2}{3s}$$

$$CV = \tfrac{1}{3}s + \frac{a^2 + b^2 - 2c^2}{3s}$$

$$BV \,.\, CV + CV \,.\, AV + AV \,.\, BV = \frac{4\Delta}{\sqrt{3}}$$

$$AV^2 + BV^2 + CV^2 = \frac{a^2 + b^2 + c^2}{2} - \frac{2\Delta}{\sqrt{3}}$$

$$AV + BV + CV = \sqrt{\left(\frac{a^2 + b^2 + c^2}{2} + 2\Delta\sqrt{3}\right)}$$

† See *Nova Acta Academiae . . . Petropolitanae* XI. 235-8 (1798)

The following are some of the properties that may be deduced from the figure consisting of a triangle and the equilateral triangles described on its sides.

(1)* $LV = BV + CV \qquad L'V' = BV' + CV'$

$MV = CV + AV \qquad M'V' = CV' + AV'$

$NV = AV + BV \qquad N'V' = AV' + BV'$

Care must be taken to affix the proper algebraic sign (+ or −) according to the position of V or V′

To prove $\quad LV = BV + CV.$

Ptolemy's theorem applied to the cyclic quadrilateral VBLC gives

$$LV \cdot BC = BV \cdot LC + CV \cdot LB$$

or $\quad LV \cdot BC = BV \cdot BC + CV \cdot BC;$

therefore $\quad LV = BV + CV$

(2)* $AL = BM = CN = AV + BV + CV$

$AL' = BM' = CN' = AV' + BV' + CV'$

with proper algebraic signs prefixed.

(3)† *The following six triangles are congruent to ABC, and the centres of their circumcircles lie all on the circumcircle of ABC ;*

$AN'M \quad ANM' \quad NBL' \quad N'BL \quad M'LC \quad ML'C$

This may be proved by rotating the triangle ABC round A through an angle of 60°, first counterclockwise, and second clockwise; then doing the same thing round B, and round C.

(4)‡ *The internal equilateral triangle described on any side cuts a side of each of the external equilateral triangles on the circumcircle of ABC.*

Let AM′ meet BL at D.

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* W. S. B. Woolhouse in the *Lady's and Gentleman's Diary* for 1865, p. 81. His proof of (1) is different from that in the text.

(1) and (2) are said to be given by Heinen, *Ueber Systeme von Kräften* (1834).

† John Turnbull in the *Lady's and Gentleman's Diary* for 1865, p. 78.

‡ Rev. William Mason and Thomas Dobson in the *Lady's and Gentleman's Diary* for 1865, pp. 76, 78.

Since $\angle CAD$ and $\angle CBD$ are each 60°,
therefore A B C D are concyclic *

(5)† $$AL^2 + AL'^2 = BM^2 + BM'^2 = CN^2 + CN'^2$$
$$= a^2 + b^2 + c^2$$

Join LL'.

Then BC LL' bisect each other perpendicularly in A';
therefore
$$AL^2 + AL'^2 = 2AA'^2 + 2A'L^2$$
$$= 2AA'^2 + 6A'B^2$$
$$= 2AA'^2 + 2A'B^2 + 4A'B^2$$
$$= AB^2 + AC^2 + BC^2$$

(6)† $$AL^2 - AL'^2 = BM^2 - BM'^2 = CN^2 - CN'^2$$
$$= 4\sqrt{3}\,\Delta = \frac{abc\sqrt{3}}{R}$$

Draw AK perpendicular to LL'.

Then
$$AL^2 - AL'^2 = LK^2 - L'K^2$$
$$= (LK + L'K)(LK - L'K)$$
$$= 2A'L \,.\, 2A'K$$
$$= 2\sqrt{3}\,A'B \,.\, 2A'K$$
$$= 4\sqrt{3}\,\Delta = \frac{abc\sqrt{3}}{R}$$

(7)‡ $$AL^2 = \tfrac{1}{2}\left(a^2 + b^2 + c^2 + \frac{abc\sqrt{3}}{R}\right)$$
$$AL'^2 = \tfrac{1}{2}\left(a^2 + b^2 + c^2 - \frac{abc\sqrt{3}}{R}\right)$$

(8)§ $$AL^2 = a^2 + b^2 - 2ab\cos(C + 60°)$$
$$AL'^2 = a^2 + b^2 - 2ab\cos(C - 60°)$$

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* D will be used for a different point in (9).

† Rev. William Mason in the *Lady's and Gentleman's Diary* for 1865, p. 75. His proof is different from that in the text.

‡ Rev. William Mason in the *Lady's and Gentleman's Diary* for 1865, p. 74

§ The first of these expressions is given in T. S. Davies's edition of Hutton's *Course of Mathematics*, I. 470 (1841). It is said to occur also in Heinen, *Ueber Systeme von Kräften* (1834)

(9)* *If* AL BM CN *meet* BC CA AB *at* D E F

then
$$\frac{1}{VD}+\frac{1}{VE}+\frac{1}{VF}=\frac{2}{AV}+\frac{2}{BV}+\frac{2}{CV}$$

For triangles BVD LVC are similar;

therefore $\quad BV : VD = LV : CV$;

therefore $\quad BV \cdot CV = VD \cdot LV$

$$= VD(BV+CV);$$

therefore
$$\frac{1}{VD} = \frac{1}{BV}+\frac{1}{CV}.$$

Similarly
$$\frac{1}{VE} = \frac{1}{CV}+\frac{1}{AV}$$

and
$$\frac{1}{VF} = \frac{1}{AV}+\frac{1}{BV};$$

whence the required result follows.

A corresponding result is true for the point V′ but care must be taken to prefix the proper signs.

(10)†
$$AL^2 \cdot AL'^2 = (a^2-b^2)(a^2-c^2)+(b^2-c^2)^2$$

with corresponding values for $BM^2 \cdot BM'^2$ $CN^2 \cdot CN'^2$.

From (7) there is obtained

$$AL^2 \cdot AL'^2 = \left(\frac{a^2+b^2+c^2}{2}\right)^2 - \tfrac{3}{4}\cdot\frac{a^2b^2c^2}{R^2}$$
$$= \tfrac{1}{4}(\ a^4+b^4+c^4+2b^2c^2+2c^2a^2+2a^2b^2)$$
$$-\tfrac{3}{4}(-a^4-b^4-c^4+2b^2c^2+2c^2a^2+2a^2b^2)$$
$$= a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2$$

(11)‡
$$AL \cdot AV = \frac{b^2+c^2-a^2}{2}+\frac{abc}{2R\sqrt{3}}$$
$$AL' \cdot AV' = \frac{b^2+c^2-a^2}{2}+\frac{abc}{2R\sqrt{3}}$$

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* W. H. Levy in the *Lady's and Gentleman's Diary* for 1855, p. 71

† W. S. B. Woolhouse in the *Lady's and Gentleman's Diary* for 1865, p. 81.

‡ (11)—(14). Rev. William Mason in the *Lady's and Gentleman's Diary* for 1865, pp. 74, 75.

For $$AV = AB\frac{\sin ABM}{\sin 120^\circ} = \frac{2c}{\sqrt{3}}\sin(A + 60^\circ)\frac{b}{BM}$$

$$AV' = AB\frac{\sin ABM'}{\sin 60^\circ} = \frac{2c}{\sqrt{3}}\sin(A - 60^\circ)\frac{b}{BM'};$$

and $$AL = BM \quad AL' = BM';$$

whence, after making the necessary substitutions, the results follow.

(12) $$AL^2 . AL'^2 . VV'^2 = 3a^2b^2c^2 - \frac{a^2b^2c^2}{3R^2}(a^2 + b^2 + c^2)$$

For $$-2AL . AL'\cos VAV' = AL^2 + AL'^2 - LL'^2$$
$$= a^2 + b^2 + c^2 - 3a^2$$
$$= b^2 + c^2 - 2a^2;$$

therefore $$AL^2 . AL'^2 . VV'^2$$
$$= (AL . AV)^2AL'^2 + (AL' . AV')^2AL^2 + (AL . AV)(AL' . AV')(b^2 + c^2 + 2a^2)$$
$$= 3a^2b^2c^2 - \frac{a^2b^2c^2}{3R^2}(a^2 + b^2 + c^2)$$

(13) $$AL^2 . OV^2 = R^2\frac{a^2 + b^2 + c^2}{2} - \frac{a^2b^2c^2}{6R^2} + \frac{R\sqrt{3}}{2abc}AL^2 . AL'^2 . VV'^2$$

$$AL'^2 . OV'^2 = R^2\frac{a^2 + b^2 + c^2}{2} - \frac{a^2b^2c^2}{6R^2} - \frac{R\sqrt{3}}{2abc}AL^2 . AL'^2 . VV'^2$$

The investigation is too long to be inserted here; the initial steps are

$$2AL . AO\cos OAV = AL^2 + AO^2 - OL^2$$
$$= AL^2 + R^2 - \left(R\cos A + \frac{\sqrt{3}}{2}a\right)^2;$$

and $$AL^2 . OV^2 = AL^2(AO^2 + AV^2 - 2AO . AV\cos OAV);$$

whence, by substitution, the first result follows.

(14) $$OV^2 + OV'^2 + VV'^2 = 2R^2$$

Multiply the first equality in (13) by AL'^2

„ second „ (13) „ AL^2;

add, and make use of the equalities in (5), (6);

then $AL^2 . AL'^2(OV^2 + OV'^2 + VV'^2)$
$$= \tfrac{1}{2}R^2(a^2 + b^2 + c^2) - \tfrac{3}{2}a^2b^2c^2$$
$$= 2R^2 . AL^2 . AL'^2$$

(15)* $$AV^2 = \frac{b^2+c^2-a^2}{3} + \frac{(a^2-b^2)(a^2-c^2)}{(a^2-b^2)(a^2-c^2)+(b^2-c^2)^2} \cdot \frac{AL'^2}{3}$$

$$AV'^2 = \frac{b^2+c^2-a^2}{3} + \frac{(a^2-b^2)(a^2-c^2)}{(a^2-b^2)(a^2-c^2)+(b^2-c^2)^2} \cdot \frac{AL^2}{3}$$

These are obtained from (10) and (11)

(16) $$AV^2 + BV^2 + CV^2 = \frac{a^2+b^2+c^2}{3} + \frac{AL'^2}{3}$$
$$= \frac{a^2+b^2+c^2}{2} - \frac{abc}{2R\sqrt{3}}$$
$$AV'^2 + BV'^2 + CV'^2 = \frac{a^2+b^2+c^2}{3} + \frac{AL^2}{3}$$
$$= \frac{a^2+b^2+c^2}{2} + \frac{abc}{2R\sqrt{3}}$$

For $$(a^2-b^2)(a^2-c^2) + b^2-c^2)(b^2-a^2) + (c^2-a^2)(c^2-b^2)$$
$$= (a^2-b^2)(a^2-c^2) + (b^2-c^2)^2$$

(17) $$\Sigma(AV^2) + \Sigma(AV'^2) = a^2+b^2+c^2$$

(18) *If R S T be the circumcentres of LBC MCA NAB and R' S' T' be the circumcentres of $L'BC$ $M'CA$ $N'AB$; then LR' MS' NT' are diameters of the first triad of circles and $L'R$ $M'S$ $N'T$ are diameters of the second triad of circles.*

For $\angle BR'C = 120°$;
therefore R′ is on the circumcircle of LBC;
and LR′ bisects BC perpendicularly.

(19)† *The triangles RST $R'S'T'$ are equilateral.*

Since AV BV CV make with each other angles of 120°, and ST TR RS are respectively perpendicular to them, therefore ST TR RS make with each other angles of 60°, and therefore form an equilateral triangle.‡

* (15)—(17). W. S. B. Woolhouse in the *Lady's and Gentleman's Diary* for 1865, pp. 84, 82. See the reference to Fuss on p. 102.

† Dr Rutherford in the *Ladies' Diary* for 1825, p. 47. Probably, however, the theorem dates farther back.

‡ Prof. Uhlich ascribes this method to Kunze.

Similarly for R′S′T′.

Or thus: *

If AT, AS be joined,

$$\angle TAS = 60° + \angle BAC = \angle NAC.$$

Now AT AS are corresponding lines in the similar triangles ANB ACM,

therefore $AN : AC = AT : AS$;

therefore triangles CAN SAT are similar;

therefore $CN : ST = CA : SA$

$= \sqrt{3} : 1$;

therefore $\sqrt{3}\,ST = CN$

Values equal to this are in like manner found for TR RS;
therefore triangle RST is equilateral.

Similarly $\sqrt{3}\,S'T' = CN'$ and R′S′T′ is equilateral.

(20)† *The sum of the squares of the sides of triangles RST R′S′T′ is equal to* $a^2 + b^2 + c^2$.

For $3(ST^2 + S'T'^2) = CN^2 + CN'^2$

$= a^2 + b^2 + c^2$;

therefore $3(ST^2 + S'T'^2 + TR^2 + T'R'^2 + RS^2 + R'S'^2)$

$= 3(a^2 + b^2 + c^2)$

(21) ‡ *The difference of the areas of triangles RST R′S′T′ is equal to the area of ABC; and the sum of the areas of RST R′S′T′ is the arithmetic mean of the three equilateral triangles on BC CA AB.*

For $RST = ST^2 \frac{\sqrt{3}}{4}$

$R'S'T' = S'T'^2 \frac{\sqrt{3}}{4}$

* This is substantially the mode of proof given in the *Ladies' Diary* for 1826, p. 38.

† Dr John Casey. See his *Euclid*, p. 264 (2nd ed., 1884)

‡ Ascribed by Professor Uhlich to Féaux, Arnsberg Programm, p. 4 (1873).

therefore $$\text{RST} - \text{R'S'T'} = (\text{S T}^2 - \text{S' T'}^2)\frac{\sqrt{3}}{4}$$

$$= \tfrac{1}{3}(\text{CN}^2 - \text{CN'}^2)\frac{\sqrt{3}}{4}$$

$$= \tfrac{1}{3} \,.\, 4\sqrt{3}\,\Delta \,.\, \frac{\sqrt{3}}{4}.$$

$$\text{RST} + \text{R'S'T'} = \tfrac{1}{3}(\text{CN}^2 + \text{CN'}^2)\frac{\sqrt{3}}{4}$$

$$= \tfrac{1}{3}(a^2 + b^2 + c^2)\,\frac{\sqrt{3}}{4}.$$

(22)* *Triangles RST R'S'T' are homologous, and O the circumcentre of ABC is their centre of homology.*

For RR' bisects BC perpendicularly;
therefore RR' passes through O.
Similarly for SS' and TT'.

(23)† *The equilateral triangles RST R'S'T' have the same centroid as ABC, and their circumcircles pass respectively through V' and V.*

The perpendiculars from R S T on BC are

$$-\frac{a}{\sqrt{3}} \qquad \frac{\text{AX}}{2} + \frac{b\cos\text{C}}{\sqrt{3}} \qquad \frac{\text{AX}}{2} + \frac{c\cos\text{B}}{\sqrt{3}}$$

where AX is the perpendicular from A to BC.

Now $$b\cos\text{C} + c\cos\text{B} = a\,;$$

therefore the sum of these perpendiculars is equal to AX.
Hence the perpendicular on BC from the centroid of RST is equal to $\tfrac{1}{3}$AX;
and similarly for the other perpendiculars.
The centroid therefore of RST, and in like manner of R'S'T', is the centroid of ABC

* Stated by Reuschle in Schlömilch's *Zeitschrift*, xi. 482 (1866).

† (23)—(29) W. S. B. Woolhouse in the *Lady's and Gentleman's Diary* for 1865, pp. 86, 83, 84.

Again $\angle AV'B = 60^\circ$;

therefore V′ lies on the circumcircle of N′AB.

Now N′T is a diameter of this circle;

therefore $\angle N'V'T = 90^\circ$.

Similarly $\angle M'V'S = 90^\circ$;

therefore
$$\angle SV'T = \angle M'V'N' = \angle BV'C = \angle SRT;$$

therefore V′ lies on the circumcircle of RST.

(24) *If G be the centroid of ABC then*

$$GV = \tfrac{1}{3}AL' \quad GV' = \tfrac{1}{3}AL.$$

For
$$3GV^2 = (AV^2 + BV^2 + CV^2) - (AG^2 + BG^2 + CG^2)$$
$$= \left(\frac{a^2+b^2+c^2}{2} - \frac{abc}{2R\sqrt{3}}\right) - \frac{a^2+b^2+c^2}{3}$$
$$= \frac{a^2+b^2+c^2}{6} - \frac{abc}{2R\sqrt{3}} = \frac{AL'^2}{3}$$

(25)
$$GO^2 + GV^2 + GV'^2 = R^2$$

This follows from (5)

(26) *GV GV′ are the radii of the circumcircles of R′S′T′ RST*

For by (19)

$$ST = \frac{CN}{\sqrt{3}} = \frac{AL}{\sqrt{3}} = \frac{3GV'}{\sqrt{3}} = \sqrt{3}\,GV'$$
$$S'T' = \frac{CN'}{\sqrt{3}} = \frac{AL'}{\sqrt{3}} = \frac{3GV}{\sqrt{3}} = \sqrt{3}\,GV$$

(27) *If G′ be the centroid of the triangle OVV′*

$$GG' = \tfrac{1}{3}R$$

For
$$3GG'^2 = (GO^2 + GV^2 + GV'^2) - (G'O^2 + G'V^2 + G'V'^2)$$
$$= R^2 - \tfrac{1}{3}(OV^2 + OV'^2 + VV'^2)$$
$$= R^2 - \tfrac{2}{3}R^2 = \tfrac{1}{3}R^2$$

(28) $$VV' = \left(\frac{GV'}{GV} - \frac{GV}{GV'}\right) . GO$$

For $$AL^2 . AL'^2 . VV'^2 = \frac{3a^2b^2c^2}{R}\left(R^2 - \frac{a^2+b^2+c^2}{9}\right)$$

$$= (AL^2 - AL'^2)^2 . GO^2$$

therefore $$VV' = \frac{AL^2 - AL'^2}{AL \; . \; AL'} . GO$$

$$= \left(\frac{GV'}{GV} - \frac{GV}{GV'}\right) . GO$$

(29) $$OV^2 = 2(GO^2 + GV^2) - GO^2 . \frac{GV^2}{GV'^2}$$

$$OV'^2 - 2(GO^2 + GV'^3) - GO^2 . \frac{GV'^2}{GV^2}$$

From the expression for $AL^2 . OV^2$ in (13)
there may be obtained, by substitution and simplification,

$$OV^2 = R^2 - \frac{abc}{3R\sqrt{3}} + \frac{R\sqrt{3}}{3abc} AL'^2 . VV'^2$$

$$= R^2 - (GV'^2 - GV^2) + \frac{GV^2}{GV'^2 - GV^2} . VV'^2$$

$$= R^2 + GV^2 - GV'^2 + GO^2 . \frac{GV'^2 - GV^2}{GV'^2}$$

which by (28) reduces to the required form

(30)* *The area of the triangle OVV′ is*

$$\frac{1}{\sqrt{3}} . \frac{(b^2-c^2)(c^2-a^2)(a^2-b^2)}{(b^2-c^2)^2 + (c^2-a^2)^2 + (a^2-b^2)^2}$$

The investigation of this is too long to be given here.

(31)† *If H be the orthocentre of ABC, and Q be the mid-point of HG, then*

$$QV = GO . \frac{GV}{GV'}, \quad QV' = GO . \frac{GV'}{GV}$$

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* Mr Stephen Watson in the *Lady's and Gentleman's Diary* for 1865, p. 78

† (31—(36). W. S. B. Woolhouse in the *Lady's and Gentleman's Diary* for 1865, pp. 84, 85

FIGURE 29

The points H G O are collinear, and $HG = 2GO$

therefore $$HQ = GQ = GO.$$

Now $$OV^2 = 2(GO^2 + GV^2) - QV^2$$
$$OV'^2 = 2(GO^2 + GV'^2) - QV'^2$$

whence, by (29), the results follow

(32) *The points Q V V' are collinear*

and $$QV \cdot QV' = GO^2 = GQ^2$$

For $VV' = QV' - QV$ by (28)

(33) *The bisectors of the angles VGV' VHV' meet VV' at the same point*

Since $$QV : QG = QG : QV'$$

therefore triangles VQG GQV' are similar;

therefore, if Gn bisect $\angle VGV'$,

$$\angle QGn = \angle QGV + \angle VGn$$
$$= \angle QV'G + \angle V'Gn$$
$$= \angle QnG;$$

therefore $$Qn = QG.$$

But since $$QV : QH = QH : QV'$$

therefore triangles VQH HQV' are similar.

Hence the bisector of $\angle VHV'$ will meet VV' at a point n such that $Qn = QH$.

(34) $$\angle GVH + \angle GV'H = 180°$$

(35) *If with the three points H V V' the parallelogram HVH'V' be completed, then OH' = R.*

Denote the mid-point of QG or HO by m' and draw HH' the diagonal of the parallelogram meeting VV' in m.

Then $\quad OH'^2 = 4m'm^2$

$$= 2Gm^2 + 2Qm^2 - QG^2$$
$$= 2Gm^2 + 2Vm^2 + 2(Qm - Vm)(Qm + Vm) - QG$$
$$= GV^2 + GV'^2 + 2QV \,.\, QV' - QG^2$$
$$= GV^2 + GV'^2 + QG^2$$
$$= GV^2 + GV'^2 + GO^2 = R^2$$

therefore H′ lies on the circumcircle of ABC.

The position of H′ is further determined by

$$GH' = 2Qm = QV + QV' = GO \,.\, \left(\frac{GV}{GV'} + \frac{GV'}{GV}\right)$$

(36) *If GH' be produced to meet the circle again in H''*

then $$GH'' = \frac{GV.GV'}{GO}$$

For $$GH' \,.\, GH'' = R^2 - GO^2$$
$$= GV^2 + GV'^2$$

therefore $$GH'' = \frac{GV^2 + GV'^2}{GH'}$$
$$= \frac{GV.GV'}{GO}$$

(37)* *If the squares of the sides of ABC be in arithmetical progression, then $AV \;\; BV \;\; CV$ are in arithmetical progression*

Denote AV BV CV by $x \;\; y \;\; z$; then

$$a^2 = y^2 + z^2 - 2yz \cos 120° = y^2 + z^2 + yz \qquad (1)$$
$$b^2 = z^2 + x^2 - 2zx \cos 120° = z^2 + x^2 + zx \qquad (2)$$
$$c^2 = x^2 + y^2 - 2xy \cos 120° = x^2 + y^2 + xy \qquad (3)$$

Now since $a^2 + c^2 = 2b^2$, therefore

$$x^2 + 2y^2 + z^2 + xy + yz = 2(z^2 + x^2 + zx)$$

or $$y^2 + \tfrac{1}{2}(z + x)\, y - \tfrac{1}{2}(z + x)^2 = 0$$

The two roots of this quadratic in y are

$$\tfrac{1}{2}(z + x) \quad \text{and} \quad -(z + x)$$

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* Thomas Weddle in the *Mathematician* III. 111 (1848). The solution is taken from p. 165.

If the second be rejected, $y=\frac{1}{2}(z+x)$
and $x\ y\ z$ are in arithmetical progression.

The second root $-(z+x)$ is rejected because it is inconsistent with (1) and (3) unless the triangle be equilateral.
For, from (1) and (3)

$$(z-x)(x+y+z)=a^2-c^2$$

Now if $\qquad y=-(z+x) \quad \text{or} \quad x+y+z=0,$

$$a=c \quad \text{and therefore} \quad a=b=c.$$

When however the triangle is equilateral, not only is $y=\frac{1}{2}(z+x)$ admissible, but likewise $x+y+z=0$; only in the latter case it is not the three lines $x\ y\ z$ that make equal angles with each other, but two of them and the third produced. In fact, the condition

$$x+y+z=0$$

expresses the well-known theorem :

If lines be drawn from the vertices of an equilateral triangle to any point in the circumference of the circumcircle, the sum of two of these is equal to the third.

(38) *Triangles ABC RST are homologous ; and so are triangles ABC R'S'T'.*

(39) *If the circles VBC VCA VAB be described and through A B C perpendiculars be drawn to VA VB VC, these perpendiculars will form a triangle DEF whose vertices will be situated respectively on the three circles.*

FIGURE 30

(40) *Triangles DEF RST are similar and similarly situated, and V is their centre of similitude.*

For $\qquad \angle \mathrm{VBU}=90°$;
therefore VU is a diameter of the circle LBC,
and consequently passes through R.

(41) *Triangle DEF = four times RST.*

(42) *AL BM CN are equal to the perpendiculars of the triangle DEF*

For $\angle VLD = 90° = \angle VAE$;
therefore DL is parallel to EF.

(43)* *The point V is such that the sum of its distances from the vertices A B C is a minimum.*

Take any point P inside DEF, and let P_1 P_2 P_3 be its projections on the sides of DEF.

Then $PP_1 + PP_2 + PP_3 = AL = VA + VB + VC.$
But $PP_1 + PP_2 + PP_3 < PA + PB + PC$

(44) *Triangle DEF is the maximum equilateral triangle that can be circumscribed about ABC.*

The problem

About a given triangle to circumscribe the maximum equilateral triangle

was proposed by Thomas Moss in the *Ladies' Diary* for 1755 under the form

In the three sides of an equiangular field stand three trees at the distances of 10, 12, and 16 chains from one another; to find the content of the field, it being the greatest the data will admit of.

The solution given next year was

FIGURE 30

On AB, AC describe segments of circles to contain angles of 60°. Join their centres T S and through A draw EF parallel to ST. Then EC FB will meet at D and form the required triangle DEF.

In Gergonne's *Annales de Mathématiques* I. 384 (1811) there were proposed the two problems:

In any given triangle to inscribe an equilateral triangle which shall be the smallest possible

* The proof given here will be found in Steiner's *Gesammelte Werke* II. 729 (1882)

About any given triangle to circumscribe an equilateral triangle which shall be the greatest possible

It was also suggested that instead of supposing the inscribed and circumscribed triangles to be equilateral they may be supposed similar to given triangles. In this more general form the problems were solved by Rochat, Vecten and others in Vol. II. pp. 88–93 (1811 and 1812)

Their solutions were preceded by the following lemma.

Two triangles *t* and *t′* are given in species, and two other triangles T and T′ respectively similar to them are inscribed the one in the other, T′ in T for example. If T′ is the smallest of the triangles similar to *t′* which it is possible to inscribe in T, the triangle T will be the greatest of the triangles similar to *t* which it is possible to circumscribe about T′, and conversely.

FIGURE 31

Let ABC be a triangle similar to *t*, and let DEF be the smallest of all the triangles similar to *t′* which it is possible to inscribe in it. If ABC is not the greatest of the triangles similar to *t* which can be circumscribed about DEF, let A′B′C′ greater than ABC be such a triangle. Divide the sides of ABC at D′ E′ F′ as the sides of A′B′C′ are divided at D E F and form the triangle D′E′F′.

Then $$\text{ABC} : \text{A}'\text{B}'\text{C}' = \text{D}'\text{E}'\text{F}' : \text{DEF}.$$

If therefore ABC be less than A′B′C′, the triangle D′E′F′ will be less than DEF, which is contrary to the hypothesis.

To prove the converse.

Let ABC be the greatest of the triangles similar to *t* which it is possible to circumscribe about DEF. If DEF is not the smallest of all the triangles similar to *t′* which it is possible to inscribe in ABC, let D′E′F′ smaller than DEF be such a triangle. Through D E F let there be drawn three straight lines B′C′ C′A′ A′B′ making with the sides of DEF the same angles that BC CA AB make with their homologues in D′E′F′.

Then $$\text{DEF} : \text{D}'\text{E}'\text{F}' = \text{A}'\text{B}'\text{C}' : \text{ABC}$$

If therefore D′E′F′ be less than DEF, the triangle ABC will be less than A′B′C′, which is contrary to the hypothesis.

Hence the following solutions:

About ABC to circumscribe a triangle similar to def *and which shall be the greatest possible.*

FIGURE 32

On CA CB describe externally segments CEA CDB containing angles equal to *e* *d*; and let the arcs of these segments cut each other at P. Through C draw DE perpendicular to PC.

If DB EA meet at F, then DEF is the required triangle.

In ABC to inscribe a triangle similar to def, *and which shall be the smallest possible.*

FIGURE 33

About the triangle *def* circumscribe a triangle *abc* similar to ABC and the greatest possible.

Cut the sides of ABC at D E F in the same manner as those of *abc* are at *d* *e* *f*; then DEF is the required triangle.

Rochat remarks that each of these problems would in general admit of six solutions, unless it is specified beforehand to which sides of the triangle given in species the sides of the circumscribed triangle are to correspond, or to which angles of the triangle given in species the angles of the inscribed triangle are to correspond.

The two preceding problems are discussed in Lhuilier's *Élémens d'analyse géométrique*, pp. 252–5 (1809); and it may be interesting to compare the 26th lemma of the first book of Newton's *Philosophiae Naturalis Principia Mathematica* (2nd ed., 1713), which is

Trianguli specie et magnitudine dati tres angulos ad rectas totidem positione datas, quae non sunt omnes parallelae, singulos ad singulas ponere.

Newton adds as a corollary

Hinc recta duci potest cujus partes longitudine datae rectis tribus positione datis interjacebunt.

The preceding pages contain the early history of the isogonic points, as well as certain properties of them which are not well known either in this country or abroad. Recent researches on the triangle have brought several of these properties to light again, and have added a considerable number of new ones. Had time and space permitted these latter might have been stated if not proved. Room can be found only for the following references.

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Mr De Longchamps' *Journal de Mathématiques Élémentaires*,
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The article in the volume for 1887 is due to Messrs J. Koehler and J. Chapron; those in the volumes for 1889 and 1892 are due to Mr A. Boutin; those in the volume for 1893 to Mr Bernès

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