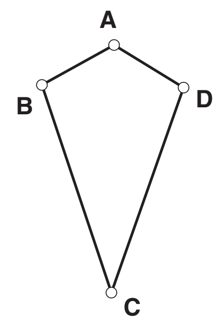
Kite Midpoints

In this investigation, you'll examine the quadrilateral formed by the midpoints of the sides of a kite. Before you begin this activity, make sure you know the properties of a kite and its diagonals.

CONJECTURE

- Open the sketch **Kite.gsp.**

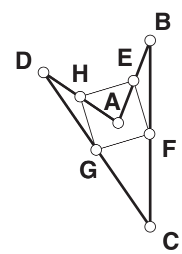


1. Drag any vertex of the quadrilateral. What features make you sure that this quadrilateral is a kite?

- Construct the midpoints of the sides of the kite.
- Connect the midpoints of the kite to construct quadrilateral *EFGH*. This is sometimes called the *midpoint quadrilateral.*

2. Drag any vertex of the kite. What kind of quadrilateral do you think *EFGH* is? Measure its angles if necessary.

- Construct diagonals $\overline{AC}$ and $\overline{BD}$.



3. What happens if kite *ABCD* becomes concave? Does your observation about the midpoint quadrilateral still hold?

- Measure the lengths of both diagonals of *ABCD.*

4. Drag any of the points *A, B, C,* and *D*. Can *EFGH* ever be a square? If so, when?

In the preceding investigation, you should have found that

- The midpoint quadrilateral of a kite is a rectangle.
- The midpoint quadrilateral of a kite is a square only when the diagonals of the kite are congruent.

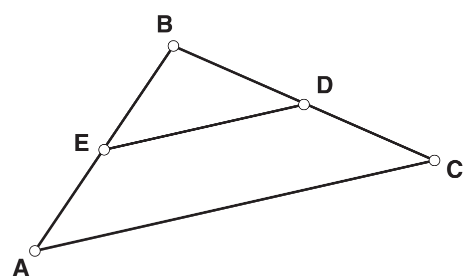
Although you are no doubt already convinced about these observations, can you *explain*, in terms of other geometric results, why your observations are true?

As before, further exploration on Sketchpad could probably succeed in convincing you even more fully, but knowing *why* something is true means understanding it much more deeply than just knowing from experimentation that it *is* true. This quest for deeper understanding is a powerful driving force not just in mathematics, but also in virtually all human intellectual pursuits.

For example, in physics, we want to understand why the planets revolve around the sun; in chemistry, why a certain chemical reacts with another, but not with some others; and in economics, why there is inflation.

EXPLAINING

Before you explain the kite midpoint quadrilateral conjectures, you'll need to make a conjecture about triangles.

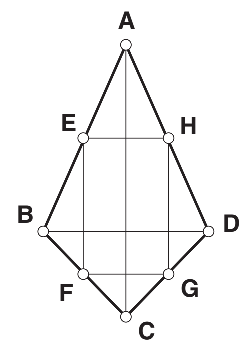


- In a new sketch, construct ΔABC.
- Construct midpoints D and E of sides BC and AB.
- Construct $\overline{DE}$. We'll call this segment a *midsegment*.
- Measure the lengths and the slopes of midsegment DE and base AC.
- Measure the ratio $\frac{ED}{AC}$.
- Drag different vertices of the triangle and observe the measures and the ratio.

5. Write a conjecture about the relationship between a midsegment and the corresponding base of its triangle.

You can use the conjecture you just made to explain why the kite midpoint conjectures are true. An explanation as to why the triangle midsegment conjecture is true can wait until later, when you explore proof as systematization in Chapter 5. For now, you can just accept the truth of the triangle midsegment conjecture.

Here are some hints for planning possible explanations of the kite midpoint conjectures. Before reading the hints, you might want to take some time to try to construct your own explanations.



6. What is the relationship between $\overline{EF}$ and $\overline{AC}$ in ΔACB? Why?

7. What is the relationship between $\overline{HG}$ and $\overline{AC}$ in ΔACD? Why?

8. What can you therefore conclude about $\overline{EF}$ and $\overline{HG}$?

9. What is the relationship between $\overline{EH}$ and $\overline{BD}$ in ΔABD? Why?

10. What is the relationship between $\overline{FG}$ and $\overline{BD}$ in ΔCBD? Why?

11. From Questions 9 and 10, what can you conclude about $\overline{EH}$ and $\overline{FG}$?

12. From Question 8 and/or Question 11, what can you conclude so far about the quadrilateral *EFGH*?

13. Given that the diagonals of a kite are always perpendicular (check if you like!), what can you now conclude about the relationships between adjacent sides of *EFGH*?

14. If $AC = BD$, what can you then say about the sides of *EFGH*?

DISCOVERING

So far we've seen that new results in mathematics can be discovered by experimentation. Sometimes, however, you can make new discoveries simply by carefully reflecting on your logical explanations. A good explanation conveys insight into why something is true and can sometimes reveal that certain conditions are not necessary and that the result is therefore merely a special case of a more general one.

15. From Question 6 to your conclusion in Question 12, did you use any properties exclusive to kites?

16. What can you therefore deduce, from your conclusion in Question 12, about *any* quadrilateral? (Make a construction to check if you like.)

17. Apart from the property of perpendicular diagonals, did you use any other property exclusive to kites for your conclusion in Question 13? (For example, did you use the property that a kite has an axis of symmetry or two pairs of adjacent sides that are equal?)

18. Use Question 17 to describe the most generic quadrilateral that always has a rectangle as its midpoint quadrilateral.

19. Apart from the function of explanation, which is shown in the other activities, what new function of a logical argument is shown in Questions 15–18?

CHALLENGE Use Sketchpad to construct the most generic quadrilateral whose midpoint quadrilateral is always a rectangle. When you succeed, describe your construction.

Present Your Explanation

Summarize your explanation from Questions 6–14 and from your further discoveries in Questions 15–19. Your summary may be in paper form or electronic form and may include a presentation sketch in Sketchpad. You may want to discuss the summary with your partner or group.

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The main purpose of this worksheet is to show the discovery function of a logical argument; that is, to show how by explaining something and identifying its underlying characteristic property, we can sometimes immediately generalize the result. You should emphasize that although the generalization in Question 16 could have been discovered by experimentation, the generalization in Question 18 could hardly have been: Who would have thought of trying a quadrilateral with perpendicular diagonals?

Prerequisites: Knowledge of properties of parallelograms, rectangles, squares, and kites.

Sketch: Kite.gsp.

CONJECTURE

1. It has (at least) one axis of symmetry through a pair of opposite angles, two pairs of adjacent sides equal, perpendicular diagonals, and so on.
2. *EFGH* is a rectangle.
3. Yes, it is still a rectangle.
4. Yes, when the diagonals are equal.

EXPLAINING

5. The midsegment of a triangle is parallel to, and equal to half of, the base of the triangle.

Note: Students should verify that the arguments below also apply to the concave case, because it is generically different (one of the diagonals falls outside).

6. $\overline{EF} \parallel \overline{AC}$ (and $EF = \frac{1}{2}AC$). (*E* and *F* are midpoints of sides *AB* and *BC*.)
7. $\overline{HG} \parallel \overline{AC}$ (and $HG = \frac{1}{2}AC$). (*H* and *G* are midpoints of sides *AD* and *DC*).
8. $\overline{EF} \parallel \overline{HG}$; $EF = HG$.

9–11. Similarly to Questions 6–8, $\overline{EF} \parallel \overline{HG}$; $EF = HG$. Let students write it out fully, but point out that for reasons of economy, it is customary to say "Similarly, it follows . . . "

12. *EFGH* is a parallelogram, since opposite sides are parallel (or one pair of opposite sides are equal and parallel).
13. Since $\overline{EF}$ and $\overline{HG}$ are parallel to $\overline{AC}$, and $\overline{EH}$ and $\overline{FG}$ are parallel to $\overline{BD}$, $\overline{AC} \perp \overline{BD}$ implies that the pairs of opposite sides of *EFGH* are all perpendicular to each other; that is, all angles are 90°. Therefore, *EFGH* is a rectangle.
14. If $AC = BD$, all the sides of *EFGH* are equal, which means it is a square.

DISCOVERING

15–16. No. *ABCD* therefore need not be a kite for *EFGH* to be a parallelogram, so this result (known as Varignon's theorem) would be true for any quadrilateral.

17–18. No. *ABCD* therefore need not be a kite for *EFGH* to be a rectangle, so this result would be true for any quadrilateral with perpendicular diagonals.

19. The discovery function of a logical explanation.

CHALLENGE First construct two perpendicular lines, and then construct points on these lines as vertices for the quadrilateral. Note that this quadrilateral is surprisingly flexible: It can be dragged into convex, concave, and crossed forms.