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Centroid of a quadrilateral

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Abstract

We give a simple rule-and-compasses construction for locating the centroid of a plane quadrilateral lamina, and we make some deductions about linear dependence in the plane.

Let $\Omega = \mathrm{PQRS}$ be a convex quadrilateral, and Ω^* the lamina $\Omega \cup \mathrm{inside}\,(\Omega)$; its area A is of course the sum of areas of two triangles. We introduce the notation A_{P} for the area of the triangle *having vertex* P *and the opposite diagonal as its opposite side*, that is $\triangle$PQS; so that

$$A = A_{\mathrm{P}} + A_{\mathrm{R}} = A_{\mathrm{Q}} + A_{\mathrm{S}}. \tag{1}$$

Let an origin O be taken anywhere in the plane not collinear with any two of the vertices, and let $\mathbf{P}, \mathbf{Q}, \ldots$ denote the position vectors of the points $\mathrm{P}, \mathrm{Q}, \ldots$ from O. Although the four vectors $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$ are linearly dependent, they form a convenient spanning set for the plane; any linear combination of them denotes a unique point in the plane, even though no point has a unique such representation. Consider the points (see Figure 1) and their vectors

$$\mathbf{M} = \tfrac{1}{2}(\mathbf{P} + \mathbf{R}), \quad \mathbf{N} = \tfrac{1}{2}(\mathbf{Q} + \mathbf{S}),$$
$$\mathbf{X} = \tfrac{1}{4}(\mathbf{P} + \mathbf{Q} + \mathbf{R} + \mathbf{S}), \quad \mathbf{W} = (A_{\mathrm{P}}\mathbf{P} + A_{\mathrm{Q}}\mathbf{Q} + A_{\mathrm{R}}\mathbf{R} + A_{\mathrm{S}}\mathbf{S})/2A. \tag{2}$$

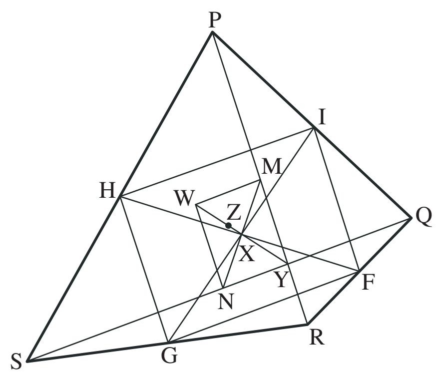

Figure 1.

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The points M and N are the midpoints of the diagonals of Ω; X is the unweighted mean of the vertices, both of Ω and of Λ (the median parallelogram of Ω), and is also the point of intersection of the diagonals of Λ. (Λ is defined to be the parallelogram FGHI whose vertices are the midpoints of the sides of Ω, taken in order; see [1].) We define Y and Z presently.

The centroid of the lamina Ω^* can be calculated as the mean of the centroids of the two triangles making up the quadrilateral, say $\triangle$PRQ and $\triangle$PRS, weighted according to the areas of these triangles. The centroid of a triangle is the common point of its medians, so the centroid of $\triangle$PRQ is $\frac{1}{3}(\mathbf{P}+\mathbf{Q}+\mathbf{R}) = (2\mathbf{M}+\mathbf{Q})/3$; thus the centroid of Ω is

$$\begin{aligned}\mathbf{Z} &= (A_{\mathrm{Q}}(\mathbf{P}+\mathbf{Q}+\mathbf{R})/3 + A_{\mathrm{S}}(\mathbf{P}+\mathbf{R}+\mathbf{S})/3)/(A_Q + A_S) \\ &= 2\mathbf{M}/3 + (A_{\mathrm{Q}}\mathbf{Q} + A_S\mathbf{S})/3A. \end{aligned} \tag{3}$$

Similarly, by regarding Ω as made up of $\triangle$PQS and $\triangle$RQS, we get

$$\mathbf{Z} = 2\mathbf{N}/3 + (A_{\mathrm{P}}\mathbf{P} + A_{\mathrm{R}}\mathbf{R})/3A. \tag{4}$$

Averaging (3) and (4) gives

$$\mathbf{Z} = \tfrac{1}{3}(\mathbf{M}+\mathbf{N}) + \tfrac{1}{3}\mathbf{W} = \tfrac{1}{3}(2\mathbf{X}+\mathbf{W}). \tag{5}$$

Lemma 1. $A_{\mathrm{R}}\mathbf{P}+A_{\mathrm{P}}\mathbf{R} = A_{\mathrm{S}}\mathbf{Q}+A_{\mathrm{Q}}\mathbf{S} = A\mathbf{Y}$, where Y is the point of intersection of the diagonals of Ω.

Proof. Let Y be the point of intersection of PR and QS. Clearly $\mathbf{Y} = (|\mathrm{YR}|\mathbf{P} + |\mathrm{YP}|\mathbf{R})/|\mathrm{PR}| = (|\triangle\mathrm{RQS}|\mathbf{P} + |\triangle\mathrm{PQS}|\mathbf{R})/A$, since the areas of these two triangles on the same base QS are to one another as their vertical heights, and hence as the lengths |YR|, |YP|. That is, $A\mathbf{Y} = A_{\mathrm{R}}\mathbf{P} + A_{\mathrm{P}}\mathbf{R}$. The second equation is proved in the same way. □

From these formulae we easily deduce the equality of the vectors $\overrightarrow{\mathrm{WM}}$ and $\overrightarrow{\mathrm{NY}}$, proving that $\Psi :=$ MYNW is a parallelogram. Also X is the midpoint of WY. Then (5) gives the location of the centroid on XW. We have established the following Construction.

Construction. To find the centroid Z of the quadrilateral lamina Ω^*:

(1) draw the diagonals of Ω and locate their point of intersection Y and their midpoints M, N;
(2) find the midpoint X of MN;
(3) find W, as the fourth vertex of the parallelogram MYNW;

then Z is the point of trisection of XW which is the closer to X.

All these steps are effected by simple use of straight-edge and compasses.

The parallelogram Ψ has sides parallel to those of Λ. Also Ψ and Λ have the same centre, but although Ψ is symmetrically placed inside Λ, it is in general not proportional to Λ.

Dart. The above discussion deals with the case when Ω is convex. Suppose instead that Ω is a dart, with its concavity at S and point at Q. See Figure 2. There are some changes to the signs of terms, but the theory is similar. Instead of (1) we have $A = A_{\mathrm{P}} + A_{\mathrm{R}} = A_{\mathrm{Q}} - A_{\mathrm{S}}$, and Y is the point of intersection of QS produced and PR. Thus in place of Lemma 1 we have $A_{\mathrm{R}}\mathbf{P} + A_{\mathrm{P}}\mathbf{R} = -A_{\mathrm{S}}\mathbf{Q} + A_{\mathrm{Q}}\mathbf{S} = A\mathbf{Y}$, and the new definition of W is

$$\mathbf{W} = (A_{\mathrm{P}}\mathbf{P} + A_{\mathrm{Q}}\mathbf{Q} + A_{\mathrm{R}}\mathbf{R} - A_{\mathrm{S}}\mathbf{S})/2A.$$

With these changes (5) holds, and the same construction applies.

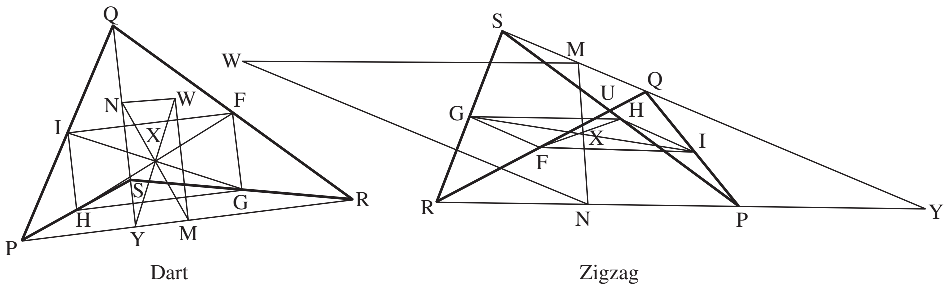

Figure 2.

Zigzag. There is now a new point U, the intersection of two opposite sides, say PS and QR. The natural adaption of the preceding theory takes A to be the *difference* of the areas of the two components of inside(Ω). As before, $A_{\mathrm{P}} = \text{area}(\triangle\text{PQS})$, etc. Assume without loss of generality that RP produced meets SQ produced in Y. Then $A = A_{\mathrm{R}} - A_{\mathrm{P}} = A_{\mathrm{S}} - A_{\mathrm{Q}}$, and now

$$A\mathbf{Y} = A_{\mathrm{R}}\mathbf{P} - A_{\mathrm{P}}\mathbf{R} = -A_{\mathrm{S}}\mathbf{Q} + A_{\mathrm{Q}}\mathbf{S},$$

$$\mathbf{W} = (-A_{\mathrm{P}}\mathbf{P} - A_{\mathrm{Q}}\mathbf{Q} + A_{\mathrm{R}}\mathbf{R} + A_{\mathrm{S}}\mathbf{S})/2A.$$

With these changes (5) holds, and the same construction applies.

Linear dependence in R^2

There must exist two independent linear relations among the four vectors $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$. Confining attention to the convex case, we get one relation from Lemma 1, namely

$$A_{\mathrm{R}}\mathbf{P} - A_{\mathrm{S}}\mathbf{Q} + A_{\mathrm{P}}\mathbf{R} - A_{\mathrm{Q}}\mathbf{S} = \mathbf{0}, \tag{6}$$

in which moreover the coefficients are all independent of the choice of origin (from their definition as areas), and also their sum is zero. One might hope that a second dependence relation could be found by using one of the zigzags PRQS or PQSR and their equivalent forms of (6), but this leads back to (6).

Let us call a nontrivial linear dependence relation *zero-sum* when the sum of the coefficients is zero; (6) is one such. We can easily prove Lemma 2.

Lemma 2. Given $\Omega = \mathrm{PQRS}$, the following properties are equivalent, for any tuple $(\alpha, \beta, \gamma, \delta)$ of reals:

(a) There exists an origin O such that $\alpha\mathbf{P} + \beta\mathbf{Q} + \gamma\mathbf{R} + \delta\mathbf{S} = \mathbf{0}$ holds as a zero-sum dependence relation on the position vectors from O, that is, where $\mathbf{P} = \overrightarrow{\mathrm{OP}}, \ldots$.

(b) For every choice of origin O′ and position vectors $\mathbf{P}' = \overrightarrow{\mathrm{O'P}}, \ldots$, the linear dependence relation $\alpha\mathbf{P}' + \beta\mathbf{Q}' + \gamma\mathbf{R}' + \delta\mathbf{S}' = \mathbf{0}'$ holds.

Thus the property of being a zero-sum relation is equivalent to the relation being independent of the choice of origin in the plane. Furthermore, we have Lemma 3.

Lemma 3. If there holds a second zero-sum linear dependence relation for Ω not equivalent to (6), then every linear dependence relation for Ω is zero-sum.

Proof. Any third linear dependence relation must be a linear combination of the given two, and therefore zero-sum. □

Take the origin O to be the centroid; then $\mathbf{Z} = \mathbf{0}$ and (5) and (2) give

$$(A + A_{\mathrm{P}})\mathbf{P} + (A + A_{\mathrm{Q}})\mathbf{Q} + (A + A_{\mathrm{R}})\mathbf{R} + (A + A_{\mathrm{S}})\mathbf{S} = \mathbf{0}.$$

This linear relation is certainly not zero-sum. By Lemma 3 we deduce Theorem 1.

Theorem 1. *Relation* (6) *is (to within a constant multiple) the sole zero-sum linear dependence relation for the convex quadrilateral* Ω.

Similar results hold for darts and zigzags. The theorem describes a general property of four position vectors in the plane, no two of them linearly dependent.

References

[1] Miller, J.B. (2007). Plane quadrilaterals. *Gaz. Aust. Math. Soc.* **34,** 103–111.