‘Symmetry’ of Cubic Functions

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As teachers, we tend not to get as much opportunity as we would like to “play” with the mathematics we teach. As often as not we get bogged down with day-to-day general school life. However, every so often something crops up that piques our curiosity to such an extent that we simply have to dig a little deeper! This happened to me recently while marking a mathematics assessment. One particular question involved determining the equation of a cubic function. The question is illustrated in Figure 1.

In the sketch, $f(x) = -x^3 + ax^2 + bx + c$.

Points $(-1; 0)$ and $(1; 4)$ are turning points.

Find the values of a, b and c.

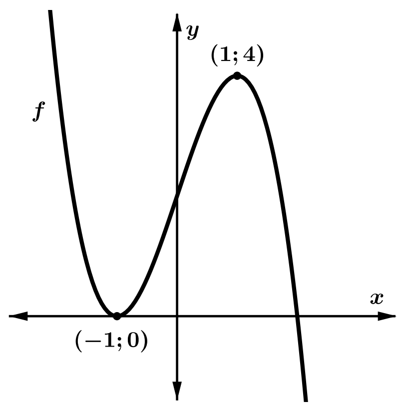

FIGURE 1: The original question.

There are a number of different ways one could determine the equation of the graph from the given information, using some combination of the facts that $f(-1) = 0$, $f(1) = 4$, $f'(-1) = 0$, $f'(1) = 0$ and, given that the point of inflection is midway between the two turning points, $f(0) = 2$ and $f''(0) = 0$. From this it can be shown that $a = 0$, $b = 3$ and $c = 2$.

As I was marking a pile of scripts I came across a pupil who had approached the question somewhat differently. Based on the shape of the graph along with the given coordinates, the pupil had made the assumption that the unknown positive x-intercept was at $(2; 0)$. Since the graph had been drawn to scale, and there was a reasonable likelihood that unknown intercept was at an integer value, this seems like a good guess. Based on this assumption, one could then write the equation of the graph in intercept form, i.e. $f(x) = -(x + 1)^2(x - 2)$ and then expand and simplify to determine the values of a, b and c.

While the assumption itself was unsubstantiated, even though it turned out to be correct, what piqued my curiosity was that the horizontal distance between the first turning point and the point of inflection (1 unit) was the same as the horizontal distance between the point of inflection and the second turning point, as well as the horizontal distance between the second turning point and the other x-intercept. Was this purely coincidental or was there something more at play here? I decided to investigate.

Let us consider a general cubic function with one turning point lying on the x-axis at $x = p$, and a further x-intercept at $x = q$. Such a cubic function would be of the form $f(x) = a(x-p)^2(x-q)$, as illustrated in Figure 2.

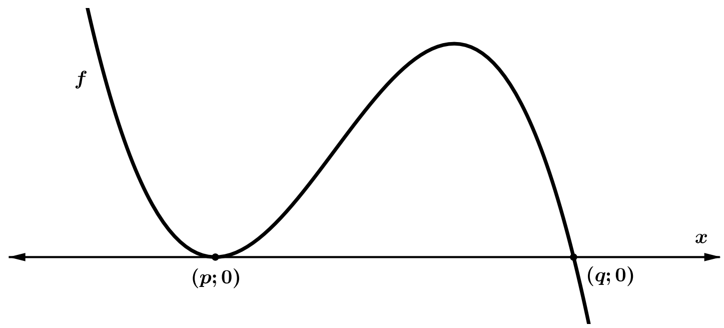

FIGURE 2: A general cubic function of the form $f(x) = a(x-p)^2(x-q)$.

We can differentiate using the product and chain rules:

$$\begin{aligned} f(x) &= a(x-p)^2(x-q) \\ f'(x) &= (x-q).2a(x-p).1 + a(x-p)^2.1 \\ &= a(x-p)[2(x-q)+(x-p)] \\ &= a(x-p)[3x-2q-p] \end{aligned}$$

The first derivative will equal zero when $x - p = 0$ or when $3x - 2q - p = 0$. The graph thus has turning points at $x = p$ and $x = \frac{2q+p}{3}$. The first solution is unsurprising as this was one of our initial conditions. At first glance the second solution seems insignificant. However, the horizontal distance between this point and the closest x-intercept is:

$$q - \frac{2q+p}{3} = \frac{3q-(2q+p)}{3} = \frac{q-p}{3}$$

This is an interesting result as it means that the horizontal distance between the turning point not on the x-axis and the nearest x-intercept will always be one third of the distance between the two x-intercepts. This, coupled with the fact that the point of inflection lies midway between the two turning points means that for functions of the form $f(x) = a(x-p)^2(x-q)$, the region between the two x-intercepts is in effect subdivided into three equally spaced parts, as illustrated in Figure 3.

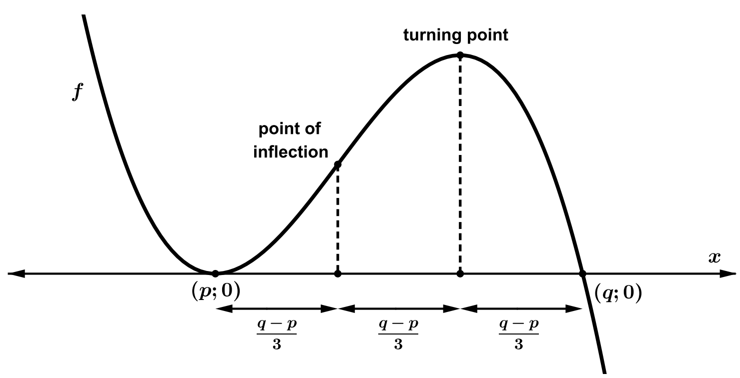

FIGURE 3: Subdividing the region between the turning points into thirds.

Having taught calculus and cubic graphs for many years I was surprised that I had never come across this result before. I assumed it was a well-known result and that I had simply never noticed it before. I posed the question to the broader mathematical community via the Google Groups "South African FET Mathematics" group asking if anyone had come across this before. I was fully expecting the response to be "yes, everyone knows that!". This, however, was not the case, and some interesting discussion arose from my initial post.

Combining the responses of Graeme Evans and Alan Christison, looking at a special case where the turning point on the x-axis occurs at the origin, i.e. $f(x) = -x^2(x - p)$, then the subdivision into thirds can readily be seen by considering the first and second derivatives of the function. This is illustrated in Figure 4.

$$f(x) = -x^3 + px^2$$

$f'(x) = -3x^2 + 2px \rightarrow$ at the turning point: $-x(3x - 2p) = 0 \rightarrow x = \frac{2}{3}p$

$f''(x) = -6x + 2p \rightarrow$ at the point of inflection: $-6x + 2p = 0 \rightarrow x = \frac{1}{3}p$

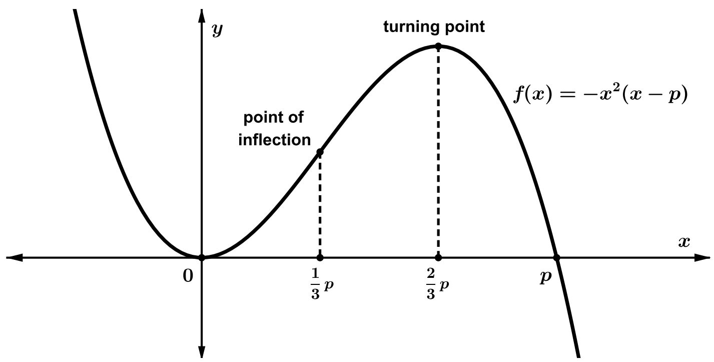

FIGURE 4: Considering a special case.

While this fairly simple result is only applicable to a very specific set of cubic functions, what intrigued me was that so few people seemed to have come across the result before. This serendipitous discovery arose from a simple assumption made by a pupil in an assessment, an assumption that I set out to disprove, and that on closer inspection revealed an interesting generality. It's a sobering and exciting reminder that no matter how many years one has been teaching for, there is always more that one can discover through observation and exploration.