e.g. a torus. There is still a shortest and a longest distance from P to a torus C, both segments being perpendicular to C. In addition we find extrema of different types representing maxima of minima or minima of maxima. To find them, we draw on the torus a closed "meridian" circle L, as in Figure 195, and we seek on L the point Q nearest to P. Then we try to move L so that the distance PQ becomes: a) a minimum. This Q is simply the point on C nearest to P. b) a maximum. This yields another stationary point. We could just as well seek on L the point farthest from P, and then find L such that this maximum distance is: c) a maximum, which will be attained at the point on C farthest from P. d) a minimum. Thus we obtain four different stationary values of the distance.

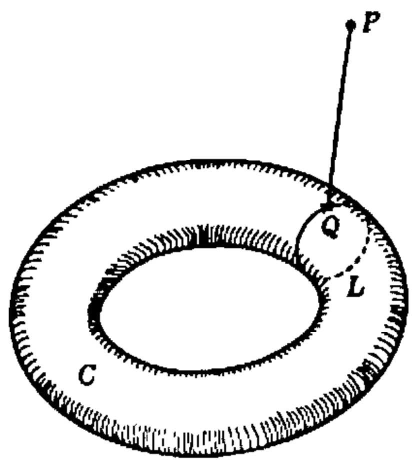

Fig. 195

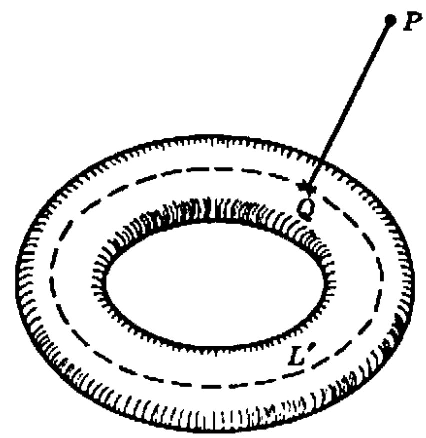

Fig. 196

* *Exercise:* Repeat the reasoning with the other type L' of closed curve on C that cannot be contracted to a point, as in Figure 196.

§4. SCHWARZ'S TRIANGLE PROBLEM

1. Schwarz's Proof

Hermann Amandus Schwarz (1843–1921) was a distinguished mathematician of the University of Berlin and one of the great contributors to modern function theory and analysis. He did not disdain to write on elementary subjects, and one of his papers treats the following problem: Given an acute-angled triangle, to inscribe in it another triangle with the least possible perimeter. (By an inscribed triangle we mean one with a vertex on each side of the original triangle.) We shall see that there is exactly one such triangle, and that its vertices are the foot-points of the altitudes of the given triangle. We shall call this triangle the *altitude triangle*.

Schwarz proved the minimum property of the altitude triangle by the method of reflection, with the help of the following theorem of elementary geometry (see Fig. 197): At each vertex, P, Q, R, the two sides of the altitude triangle make equal angles with the side of the original triangle; this angle is equal to the angle at the opposite vertex of the original triangle. For example, the angles ARQ and BRP are both equal to angle C, etc.

To prove this preliminary theorem, we note that $OPBR$ is a quadrilateral that can be inscribed in a circle, since $\measuredangle OPB$ and $\measuredangle ORB$ are right angles. Consequently, $\measuredangle PBO = \measuredangle PRO$, since they subtend the same arc PO in the circumscribed circle. Now $\measuredangle PBO$ is complementary to $\measuredangle C$, since CBQ is a right triangle, and $\measuredangle PRO$ is complementary to $\measuredangle PRB$. Therefore the latter is equal to $\measuredangle C$. In the same way, using the quadrilateral $QORA$, we see that $\measuredangle QRA = \measuredangle C$, etc.

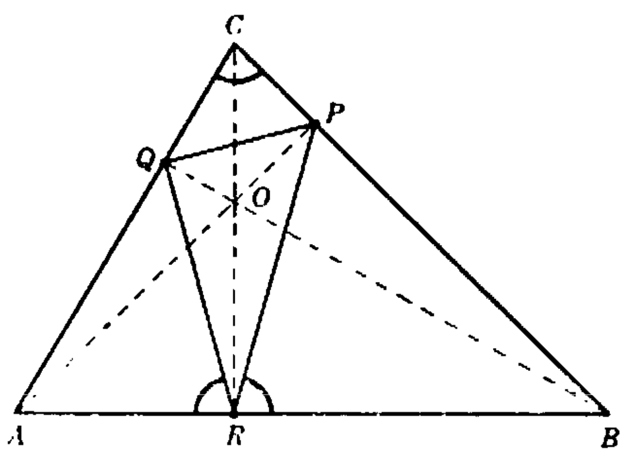

Fig. 197. Altitude triangle of ABC, showing equal angles.

This result enables us to state the following reflection property of the altitude triangle: Since, for example $\measuredangle AQR = \measuredangle CQP$, the reflection of RQ in the side AC is the continuation of PQ, and vice versa; similarly for the other sides.

We shall now prove the minimum property of the altitude triangle. In the triangle ABC consider, together with the altitude triangle, any other inscribed triangle, UVW. Reflect the whole figure first in the side AC of ABC, then reflect the resulting triangle in its side AB then in BC then again in AC, and finally in AB. In this way we obtain altogether six congruent triangles, each with the altitude triangle and the other one inscribed. The side BC of the last triangle is parallel to the original side BC. For in the first reflection, BC is rotated clockwise through an angle $2C$, then through $2B$ clockwise; in the third reflection it is not affected, in the fourth it rotates through $2C$ counterclockwise,

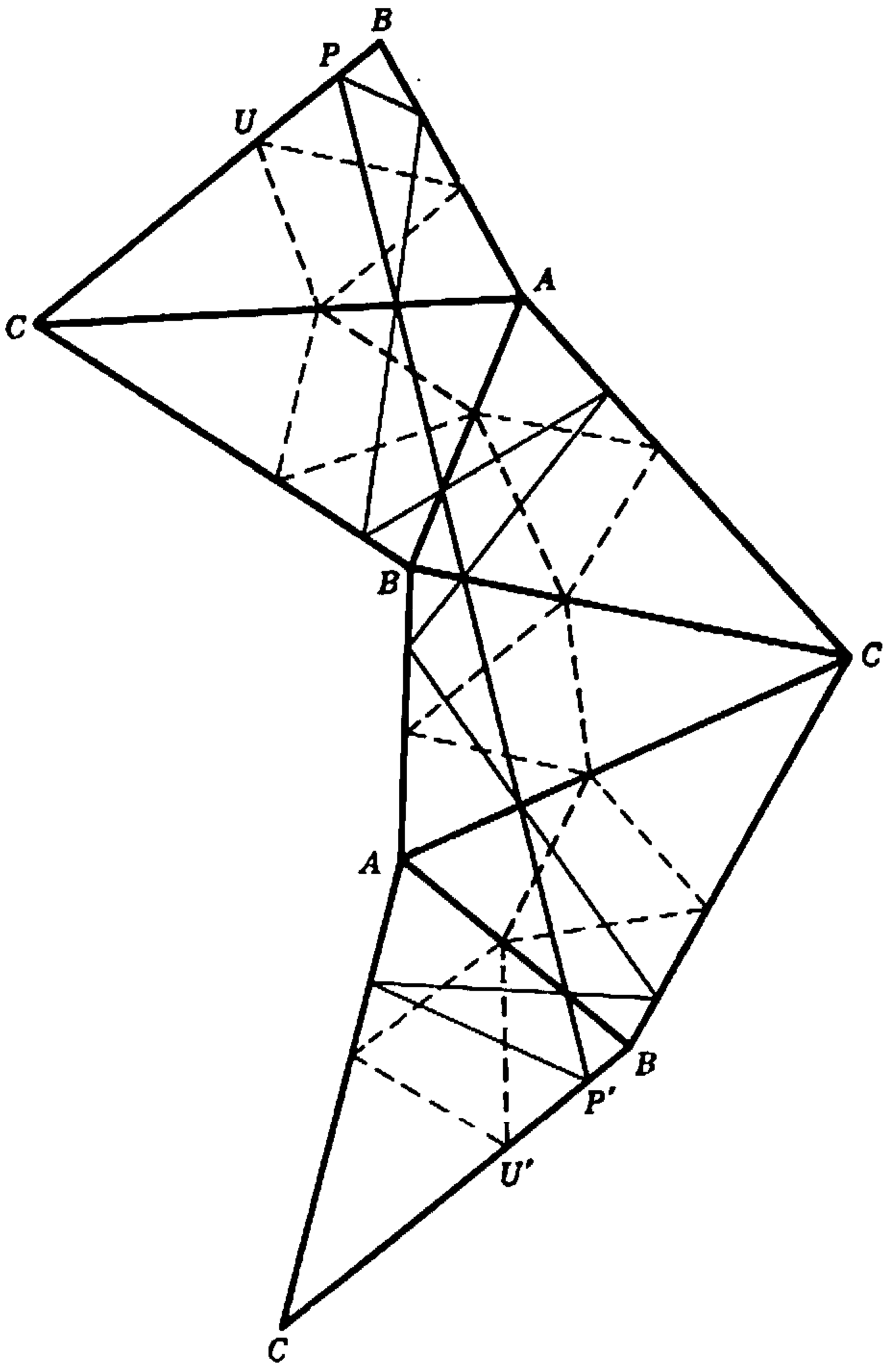

Fig. 198. Schwarz's proof that altitude triangle has least perimeter.

and in the fifth through $2B$ counterclockwise. Thus the total angle through which it has turned is zero.

Due to the reflection property of the altitude triangle, the straight line segment PP' is equal to twice the perimeter of the altitude triangle; for PP' is composed of six pieces that are, in turn, equal to the first, second, and third side of the triangle, each side occurring twice. Similarly, the broken line from U to U' is twice the perimeter of the other inscribed triangle. This line is not shorter than the straight line segment UU'. Since UU' is parallel to PP', the broken line from U to U' is not shorter than PP', and therefore the perimeter of the altitude triangle is the shortest possible for any inscribed triangle, as was to be proved. Thus we have at the same time shown that there is a minimum and that it is given by the altitude triangle. That there is no other triangle with perimeter equal to that of the altitude triangle will be seen presently.

2. Another Proof

Perhaps the simplest solution of Schwarz's problem is the following, based on the theorem proved earlier in this chapter that the sum of the distances from two points P and Q to a line L is least at that point R of L where PR and QR make the same angle with L, provided that P and Q lie on the same side of L and neither lies on L. Assume that the triangle PQR inscribed in the triangle ABC solves the minimum problem. Then R must be the point on the side AB where $p + q$ is a minimum, and therefore the angles ARQ and BRP must be equal; similarly, $\measuredangle AQR = \measuredangle CQP$, $\measuredangle BPR = \measuredangle CPQ$. Thus the minimum triangle, if it exists, must have the equal-angle property used in Schwarz's proof. It remains to be shown that the only triangle with this property is the altitude triangle. Moreover, since in the theorem on which this proof is based it is assumed that P and Q do not lie on AB, the proof does not hold in case one of the points P, Q, R is a vertex of the original triangle (in which case the minimum triangle would degenerate into twice the corresponding altitude); in order to complete the proof we must show that the perimeter of the altitude triangle is shorter than twice any altitude.

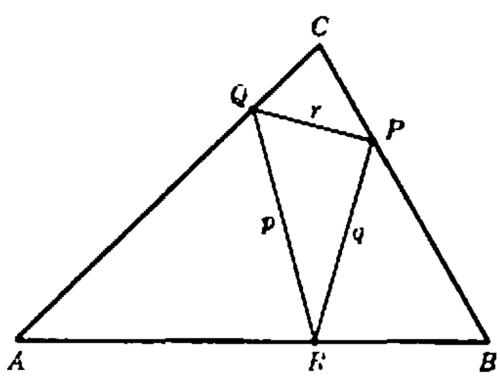

Fig. 199.

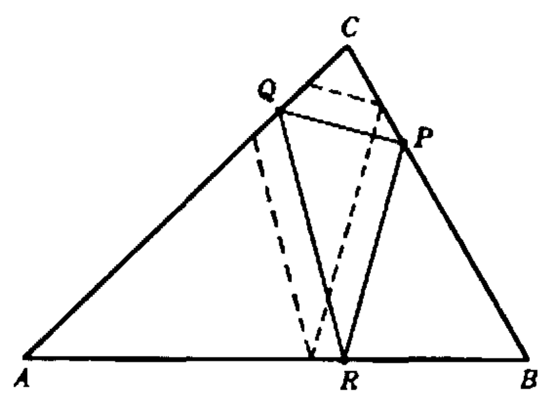

Fig. 200.

To dispose of the first point, we observe that if an inscribed triangle has the equal-angle property mentioned above, the angles at P, Q, and R must be equal to $\measuredangle A$, $\measuredangle B$, and $\measuredangle C$ respectively. For assume, say, that $\measuredangle ARQ = \measuredangle C + \delta$. Then, since the sum of the angles of a triangle is 180°, the angle at Q must be $B - \delta$, and at P, $A - \delta$, in order that the triangles ARQ and BRP may have the sum of their angles equal to 180°. But then the sum of the angles of the triangle CPQ is $A - \delta + B - \delta + C = 180° - 2\delta$; on the other hand, this sum must be 180°. Therefore δ is equal to zero. We have already seen that the altitude triangle has this equal-angle property. Any other triangle with this property would have its sides parallel to the corresponding sides of the altitude triangle; in other words, it would have to be similar to it and oriented in the same way. The reader may show that no other such triangle can be inscribed in the given triangle (see Fig. 200).

Finally, we shall show that the perimeter of the altitude triangle is less than twice any altitude, provided the angles of the original triangle are all acute. We produce the sides QP and QR and draw the

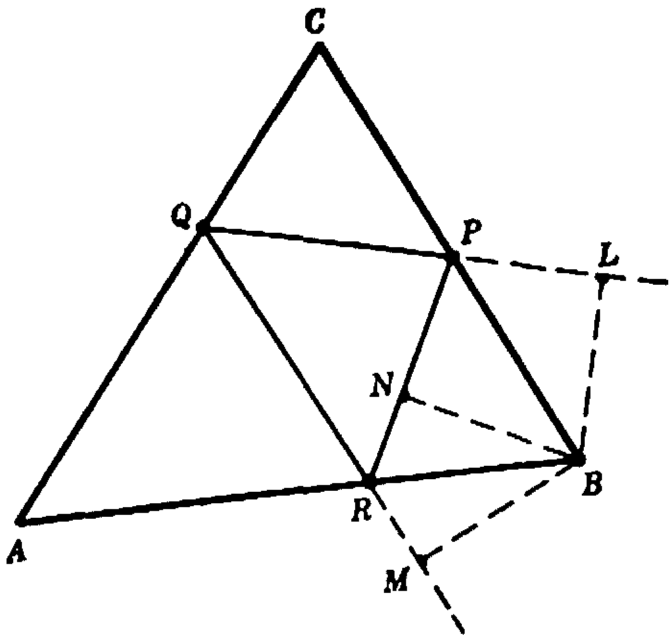

Fig. 201.

perpendiculars from B to QP, QR, and PR, thus obtaining the points L, M, and N. Then QL and QM are the projections of the altitude QB on the lines QP and QR respectively. Consequently, $QL + QM < 2QB$. Now $QL + QM$ equals p, the perimeter of the altitude triangle. For triangles MRB and NRB are congruent, since angles MRB and NRB are equal, and the angles at M and N are right angles. Hence $RM = RN$; therefore $QM = QR + RN$. In the same way, we see that $PN = PL$, so that $QL = QP + PN$. We therefore have $QL + QM = QP + QR + PN + NR = QP + QR + PR = p$. But we have shown that $2QB > QL + QM$. Therefore p is less than twice the altitude QB; by exactly the same argument, p is less than twice any altitude, as was to be proved. The minimum property of the altitude triangle is thus completely proved.

Incidentally, the preceding construction permits the direct calculation of p. We know that the angles PQC and RQA are equal to B, and therefore $PQB = RQB = 90° - B$, so that $\cos (PQB) = \sin B$. Therefore, by elementary trigonometry, $QM = QL = QB \sin B$, and $p = 2QB \sin B$. In the same way, it can be shown that $p = 2PA \sin A = 2RC \sin C$. From trigonometry, we know that $RC = a \sin B = b \sin A$, etc., which gives $p = 2a \sin B \sin C = 2b \sin C \sin A = 2c \sin A \sin B$. Finally, since $a = 2r \sin A$, $b = 2r \sin B$, $c = 2r \sin C$, where r is the radius of the circumscribed circle, we obtain the symmetrical expression, $p = 4r \sin A \sin B \sin C$.

3. Obtuse Triangles

In both of the foregoing proofs it has been assumed that the angles A, B, and C are all acute. If, say, C is obtuse, as in Figure 202, the

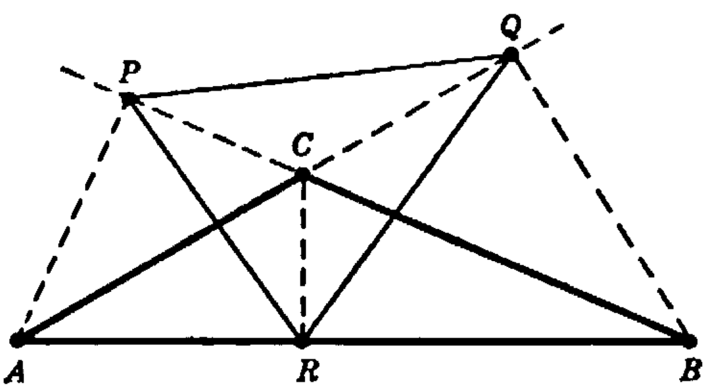

Fig. 202. Altitude triangle for obtuse triangle.

points P and Q will lie outside the triangle. Therefore the altitude triangle can no longer, strictly speaking, be said to be *in*scribed in the triangle, unless by an inscribed triangle we merely mean one whose vertices are on the sides or on the extensions of the sides of the original triangle. At any rate, the altitude triangle does not now give the minimum perimeter, for $PR > CR$ and $QR > CR$; hence $p = PR + QR + PQ > 2CR$. Since the reasoning in the first part of the last proof showed that the minimum perimeter, if not given by the altitude triangle, must be twice an altitude, we conclude that for obtuse triangles the "inscribed triangle" of smallest perimeter is the shortest altitude counted twice, although this is not properly a triangle. Still, one can find a proper triangle whose perimeter differs from twice the altitude by as little as we please. For the boundary case, the right triangle, the two solutions—twice the shortest altitude, and the altitude triangle—coincide.

The interesting question whether the altitude triangle has any sort of extremum property for obtuse triangles cannot be discussed here. Only this much may be stated: the altitude triangle gives, not a minimum for the sum of the sides, $p + q + r$, but a stationary value of minimax type for the expression $p + q - r$, where r denotes the side of the inscribed triangle opposite the obtuse angle.

4. Triangles Formed by Light Rays

If the triangle ABC represents a chamber with reflecting walls, then the altitude triangle is the only triangular light path possible in the chamber. Other closed light paths in form of polygons are not excluded, as Figure 203 shows, but the altitude triangle is the only such polygon with three sides.

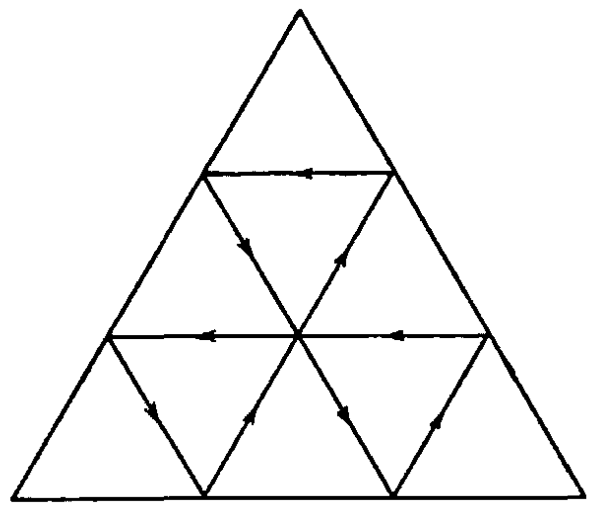

Fig. 203. Closed light path in a triangular mirror.

We may generalize this problem by asking for the possible "light triangles" in an arbitrary domain bounded by one or even several smooth curves; i.e. we ask for triangles with their vertices somewhere on the boundary curves and such that each two adjacent edges form the same angle with the curve. As we have seen in §1, the equality of angles is a condition for maximum as well as minimum total length of the two edges, so that we may, according to circumstances, find different types of light triangles. For example, if we consider the interior of a single smooth curve C, then the inscribed triangle of maximum length must be a light triangle. Or we may consider (as suggested to the authors by Marston Morse) the exterior of three smooth closed curves. A light triangle ABC may be characterized by the fact that its length has a stationary value; this value may be a minimum with respect to all three points A, B, C, it may be a minimum with respect to any of the combinations such as A and B and a maximum with respect to the third