4.5 Fagnano's problem

The properties of the mirror image can be used to derive many interesting theorems simply and in a striking fashion. We shall use these properties to solve the problem of finding the triangle of minimal perimeter inscribed in a given acute-angled triangle. This is known as Fagnano's problem†.

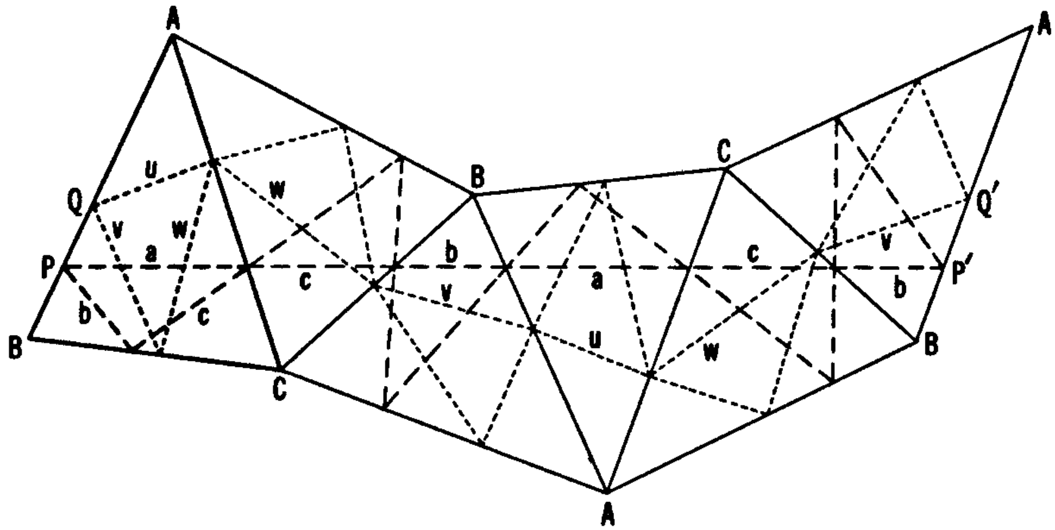

Figure 4.5A

For a solution (see Figure 4.5A), we begin with the arbitrary acute-angled triangle ABC, in which we have inscribed two triangles: the orthic triangle (dashed lines) and any other triangle (dotted lines). Let us reflect $\triangle ABC$, with contents, in its sides AC, CB, BA, AC, CB in succession. Now we inspect the diagram to see what this continued sequence of reflections has done to our triangles.

Disregarding the two points marked C, we observe a broken line $BABABA$, having angles (measured counterclockwise) $2A$ at the first point A (top left), $2B$ at the second point B (in the middle), $-2A$ at the second point A (at the bottom), and $-2B$ at the third point B (on the right). The zero sum of these four angles indicates that the final side BA is congruent *by translation* to the original side BA, and that pairs of corresponding points on these two sides will form a parallelogram such as $PP'Q'Q$.

We now recall that the altitudes of $\triangle ABC$ bisect the angles of its orthic triangle. It follows that, after the indicated reflections, the sides of the orthic triangle will, in order, lie on the straight line PP', shown in Figure 4.5A. Analogously, the sides of any other triangle, such as the

† Proposed in 1775 by Fagnano, who solved it by calculus. The proof shown here is due to H. A. Schwarz. For another proof, also using reflections, see Coxeter [**6**, p. 21] or Kazarinoff [**18**, pp. 76–77] or Courant and Robbins [**4**, p. 347]. Schwarz's treatment was extended from triangles to $(2n + 1)$-gons by Frank Morley and F. V. Morley, *Inversive Geometry* (Ginn, Boston, 1933), p. 37.

dotted triangle in the figure, will form a broken line reaching from Q (on the original AB) to Q' (on the final AB). Since PQ is equal and parallel to $P'Q'$, the straight segment QQ' is equal to PP', which is twice the perimeter of the orthic triangle. This is clearly shorter than the broken line from Q to Q', which is twice the perimeter of the other triangle. Hence the triangle of minimal perimeter is the orthic triangle.

4.6 The three jug problem

A curious application of reflection† is to the solution of problems requiring the division of a liquid into stated portions with what appear to be inadequate measuring devices. This application requires a preliminary account of trilinear coordinates, which we now present.

As a welcome relief from the ordinary squared paper, used for plotting points with given Cartesian coordinates, one can sometimes buy "triangulated" paper, ruled with three systems of parallel lines dividing the plane into a tessellation of small equilateral triangles. Such paper is convenient for plotting points that have given *trilinear* coordinates with respect to a (large) equilateral triangle. In the plane of such a triangle ABC, with side a and altitude h, the trilinear coordinates of a point P are defined to be the distances x, y, z of P from the three sides BC, CA, AB, regarded as positive when P is inside the triangle. We call P the point (x, y, z). Since

$$\begin{aligned} \tfrac{1}{2}ax + \tfrac{1}{2}ay + \tfrac{1}{2}az &= (PBC) + (PCA) + (PAB) \\ &= (ABC) = \tfrac{1}{2}ah, \end{aligned}$$

we have

$$x + y + z = h.$$

These coordinates are ideal for representing any situation in which three variable quantities have a constant sum. When one of the quantities stays fixed while the other two vary (with a constant sum), the point (x, y, z) moves along a line parallel to one side of the triangle. In particular, the sides themselves have the equations

$$x = 0, \qquad y = 0, \qquad z = 0,$$

and the vertices A, B, C have the coordinates $(h, 0, 0)$, $(0, h, 0)$, $(0, 0, h)$.

One such situation arises when h pints (or ounces) of a liquid are distributed into three vessels so that there are x pints in the first, y in the second, and z in the third. The operation of pouring liquid gradually

† M. C. K. Tweedie, *Mathematical Gazette* **23** (1939), pp. 278–282; A. I. Perel'man, Zanumatel'naya Geometria (Moscow, 1958); T. H. O'Beirne [21], pp. 49–75.