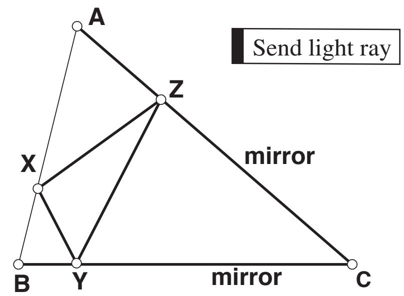


Although this problem is a purely geometric one, it will be easier if you interpret it as a problem in physics. Imagine sitting in a triangular room *ABC* with walls $\overline{BC}$ and $\overline{AC}$ that are mirrors—a little bit like sitting inside a kaleidoscope. A light ray from a laser, sent from a point *X* on $\overline{AB}$ to $\overline{BC}$, reflects from $\overline{BC}$ at *Y* to $\overline{AC}$, and reflects from $\overline{AC}$ at *Z* back to *X*. Where should the light ray start and where should it hit each wall for it to follow the shortest possible path?

CONJECTURE

- Open the sketch **Light Ray.gsp.** Press the button to send the light ray around the triangle.
- Press the buttons to show the measures of $\angle XYB$, $\angle ZYC$, $\angle YZC$, $\angle XZA$, $\angle AXZ$, and $\angle BXY$.

1. What do you notice about these angle measures? Check your observations by dragging point *X*.

2. Explain your observation from Question 1 using what you know about light rays.

- Drag point *X* along $\overline{AB}$ until the perimeter *XYZ* is a minimum.
- Press the button to show each of the three altitudes and their feet.

If two points overlap, it is possible to drag one point when you want to drag the other point. If this happens, try to select and drag the point again.

3. What do you notice about the positions of *X*, *Y*, and *Z* in relation to the feet of the altitudes?

- Drag any vertex of ΔABC to a new position, but keep the triangle acute. Again drag *X* until the perimeter of *XYZ* is a minimum and check your observation in Question 3.
- Repeat the preceding step at least one more time.

4. From Questions 1 and 3, what can you conjecture about the pairs of angles at the feet of the altitudes, such as $\angle DFA$ and $\angle EFC$, $\angle FDA$ and $\angle EDB$, and $\angle DEB$ and $\angle FEC$?

Check your conjecture in Question 4 by pressing the button to show the angle measures at the feet of the altitudes.

5. Is your conjecture in Question 4 also true if ΔABC is obtuse?

6. **Certainty:** Look back at your conjecture in Question 3 and your conjecture in Question 4. How certain are you that each conjecture is always true? Can you provide convincing proofs or counterexamples to back up your position? Record your level of certainty on the number line and explain your choice.

Conjecture in Question 3:

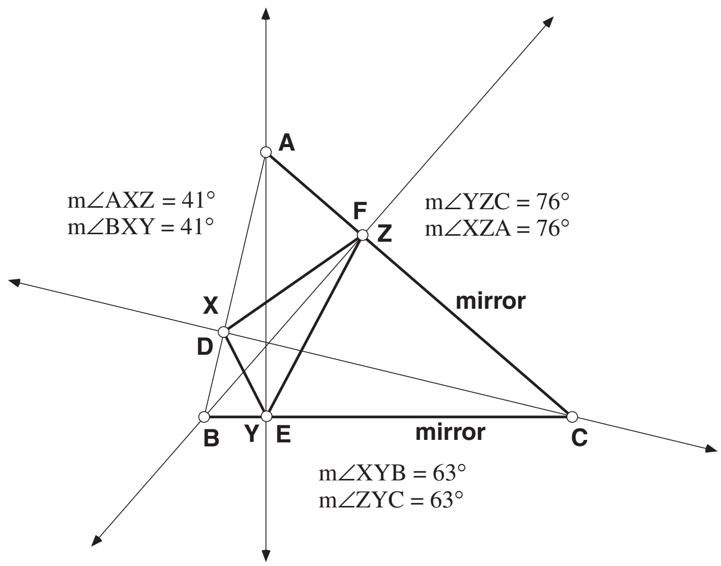
0% **25%** **50%** **75%** **100%**

Conjecture in Question 4:

0% **25%** **50%** **75%** **100%**

CHALLENGE If you believe your conjecture is always true, provide some examples to support your view and try to convince your partner or members of your group. Even better, support your conjecture with a logical explanation or a convincing proof. If you suspect your conjecture or your partner's conjecture is not always true, try to supply counterexamples.

PROVING



You have probably made these two conjectures:

- In acute triangle *ABC*, inscribed triangle *XYZ* has its minimum perimeter when its vertices lie at the feet of the altitudes.
- The pairs of angles surrounding the feet of the altitudes of triangle *ABC* are equal (for example, $m\angle DFA = m\angle EFC$, $m\angle FDA = m\angle EDB$, and $m\angle DEB = m\angle FEC$).

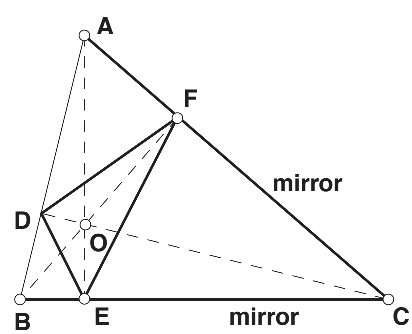
But how certain are you? As you may have seen in some earlier experiences, it is possible to draw erroneous conclusions just from observations. For example, conjectures can break down when extreme cases are considered. How do you know that you have checked all possible cases?

Work through the arguments below to convince yourself of the truth of your conjectures. You will prove the second conjecture first.

PROVING ANGLE MEASURES EQUAL

To construct an intersection point, select two lines and choose **Intersection** from the Construct menu. To change a label, double-click on the label with the **Text** tool.

Construct the intersection of the altitudes and label the intersection *O*.



7. In quadrilateral *OECF*, what can you say about opposite angles *OEC* and *OFC*? Why?

To construct an arc, select three vertices in order around the quadrilateral and then choose **Arc Through 3 Points** from the Construct menu. Do this with another set of three points.

8. Use Question 7 to explain why *OECF* is a cyclic quadrilateral (that is, a quadrilateral inscribed in a circle).

9. From Question 8, what can you conclude about $\angle EOC$ and $\angle EFC$?

10. In quadrilateral *ADOF*, what can you say about opposite angles *ADO* and *AFO*? What type of quadrilateral is *ADOF*, therefore? (Check your conclusion by a construction in Sketchpad, if you like.)

11. From Question 10, what can you conclude about $\angle AFD$ and $\angle AOD$?

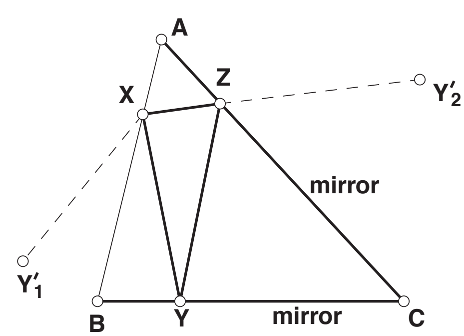
12. What can you say about $\angle EOC$ and $\angle AOD$? Why?

13. What can you therefore conclude from Questions 9, 11, and 12?

14. Explain how the same argument applies for the pairs of angles at the other two altitudes.

PROVING MINIMUM PERIMETER

Now you will prove your first conjecture. Reread your first conjecture, then work carefully through the steps that follow.



- Press the buttons to hide the altitudes and all the angle measures.
- Reflect Y across $\overline{AB}$. Call this point Y'_1.
- Reflect Y across $\overline{AC}$. Call this point Y'_2.
- Construct $\overline{Y'_1X}$ and $\overline{Y'_2Z}$.

To reflect the point, select segment or line *AB* and choose **Mark Mirror f**rom the Transform menu. Then select *Y* and choose **Reflect** from the Transform menu.

15. What can you now say about $\overline{XY'_1}$ and $\overline{XY}$, and $\overline{Y'_2Z}$ and $\overline{ZY}$? Why?

16. From Question 15, what can you say about the lengths of the path $XY + YZ + ZX$ and the path $XY'_1 + ZX + ZY'_2$?

17. What do you notice about points X, Z, and Y'_2? Try to explain (prove) your observation.

18. Drag X until the length of the path $XY'_1 + ZX + ZY'_2$ is a minimum. Explain the location of X.

19. Show (prove) that if the construction meets the condition in Question 18, then $m\angle AXZ = m\angle BXY$.

20. From Question 19 and from the result in the first proving section of this activity, what can you conclude about the position of ΔXYZ for its perimeter to be a minimum?

Presenting Your Proof

Summarize one or both of your proofs. Your summaries may be in paper form or electronic form and may include a presentation sketch in Sketchpad. You may want to discuss these summaries with your partner or group.

Further Exploration

Historical Note: The problem of an inscribed triangle with the smallest perimeter in an acute triangle was first proposed by Hermann Schwarz (1843–1921), a professor at Göttingen in Berlin, Germany, and one of the most distinguished researchers on the calculus of variations in the nineteenth century.

Use your light ray sketch to check cases where ΔABC is right or obtuse. Where should you locate ΔXYZ for it to have minimum perimeter? Try to explain your solution.

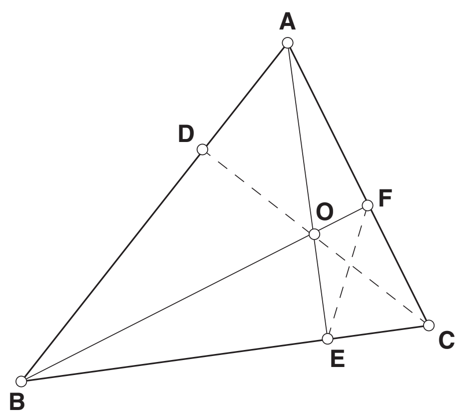
13. Yes, $\overline{BF}$ and $\overline{CD}$ are also the perpendicular bisectors of sides *GH* and *HI* of triangle *GHI*.
14. Since the perpendicular bisectors of any triangle are concurrent, $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$ are concurrent. But these lines are also the altitudes of triangle *ABC,* and are therefore concurrent.

Present Your Proof

This section provides students with the opportunity to organize the proof as a coherent whole.

Alternative Proofs

It may also be informative for students to encounter the following well-known proof for the concurrency of the altitudes. Here, two altitudes, $\overline{AE}$ and $\overline{BF}$, are drawn, and it must now be shown that the line *CD* from the remaining vertex through their point of intersection *O* is also an altitude.



OECF is cyclic, since the opposite angles at *F* and *E* measure 90°. *ABEF* is also cyclic, since $m\angle AFB = 90° = m\angle AEB$ on segment *AB*. Angle *OEF* = angle *OCF* (angles on chord *OF* of *OECF*). But angle *OEF* = angle *ABF* (angles on chord *AF* of *ABEF*). Therefore, angle *OCF* = angle *ABF,* which implies that *DBEF* is cyclic (angle *DBO* = angle *DCF* on segment *DF*). Thus, $m\angle BDO = 90°$, since it is supplementary to its opposite angle *BEO* in cyclic quad *DBOE*.

LIGHT RAY IN A TRIANGLE (PAGE 90)

This activity follows the Triangle Altitudes activity, although it can be done independently from that activity if students already know about the concurrency of the altitudes of a triangle.

Prerequisites: Knowledge of properties of cyclic quads (e.g., that equal angles on same chord or opposite angles supplementary implies that quad is cyclic).

Sketch: Light Ray.gsp.

CONJECTURE

1. Angle *XYB* = angle *ZYC* and angle *YZC* = angle *XZA,* but angles *AXZ* and *BXY* are not necessarily equal.
2. These two pairs of angles are equal because of the reflections occurring on sides *BC* and *AC.* For any reflection, the angle of incidence is equal to the angle of reflection.
3. They (appear to) coincide with the feet of the altitudes. (The triangle with minimum perimeter of an acute triangle is found at the feet of the altitudes.)
4. All three pairs of angles around the feet of the altitudes are equal.
5. Yes, it is also true.
6. Responses may vary.

CHALLENGE It is important for you, as the teacher, to take a neutral stand here, or even better that of a skeptic, and not to indicate to the students that the result is indeed true. Challenge them to convince you or other skeptics in the class.

PROVING ANGLE MEASURES EQUAL

7. The opposite angles are supplementary ($m\angle OEC = 90° = m\angle OFC$).
8. Since $\angle OEC$ and $\angle OFC$ are both right angles, they can both be inscribed in semicircles; therefore, *OECF* is a cyclic quadrilateral.
9. $m\angle EOC = m\angle EFC$ (on chord *EC*).

10. Opposite angles are supplementary ($m\angle ADO = 90° = m\angle AFO$); therefore, *ADOF* is a cyclic quad.
11. $m\angle AFD = m\angle AOD$ (on chord *AD*).
12. $m\angle EOC = m\angle AOD$, since they are directly opposite.
13. Therefore, $m\angle EFC = m\angle AFD$. (Note that their complementary angles, *BFE* and *BFD*, are therefore also equal.)
14. Responses may vary.

Notes

It is possible to come up with several different variations on the above proof, and students may find it useful to compare their efforts. It may be instructive for students to repeat the above proof for the cases in which triangle *ABC* is right or obtuse.

PROVING MINIMUM PERIMETER

15. $XY_1' = XY$ and $Y_2'Z = ZY$ from the reflections.
16. The two paths are equal in length.
17. X, Z, and Y_2' are always collinear. Explanation: $m\angle XZA = m\angle YZC$ (from the construction used to model the situation), but $m\angle YZC = m\angle Y_2'ZC$ (from the reflection around *AC*). Therefore, $m\angle XZA = m\angle Y_2'ZC$. Since *AC* is given as a straight line, $m\angle XZA + m\angle XZC = 180°$. Therefore, $m\angle Y_2'ZC + m\angle XZC = 180°$; thus, XZY_2' is also a straight line (X, Z, and Y_2' are collinear).
18. The path $XY_1' + ZX + ZY_2'$ will be a minimum when it is a straight line. Therefore, X must be positioned so that $m\angle AXZ = m\angle BXY_1'$ (vertically opposite angles must be equal).
19. $m\angle BXY_1' = m\angle BXY$ (from the reflection around *AB*). So if the condition in Question 18 ($m\angle AXZ = m\angle BXY_1'$) is met, $m\angle AXZ = m\angle BXY$.
20. For triangle *XYZ* to have minimum perimeter, the following three pairs of angles must be equal: $m\angle AXZ = m\angle BXY$, $m\angle BYX = m\angle CYZ$, and $m\angle XZA = m\angle YZC$. But from the first result, we have the three pairs of angles surrounding the feet of pedal triangle *DEF* equal; that is, $m\angle DFA = m\angle EFC$, $m\angle FDA = m\angle EDB$, and $m\angle DEB = m\angle FEC$. Therefore, triangle *XYZ* must coincide with the pedal triangle *DEF.* (Or, alternatively, pedal triangle *DEF* meets this criterion of having the angles surrounding its feet equal; therefore triangle *XYZ* must coincide with the pedal triangle *DEF.*)

Note: The above argument shows that the feet of the pedal triangle meet the criterion, and thus provides a solution, but does not show the uniqueness of this solution. A complete proof that the triangle with minimum perimeter lies only at the feet of the altitudes can be found in Hildebrandt and Tromba (1985, 60–63).