

In earlier activities, you may have discovered geometric properties by making a construction in Sketchpad and then producing a logical explanation as to why the property must hold true.

In mathematical research, experimentation does not always precede logical reasoning. As you will see in this activity, people also discover new geometric properties by logical reasoning. Only afterward do they follow up with construction and measurement to make sure that false assumptions or conclusions have not been made.

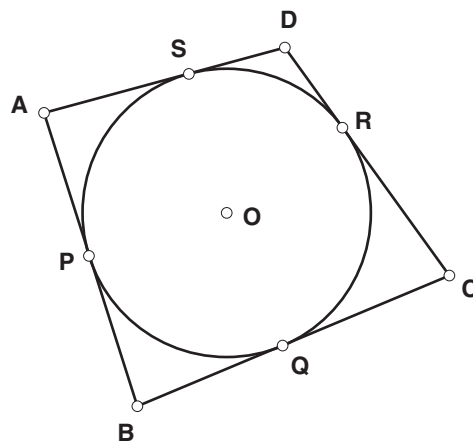
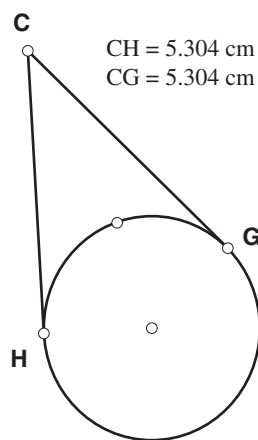
DISCOVERING



In this activity, you will logically deduce an interesting property of a quadrilateral with all its sides tangent to an inscribed circle—in other words, a quadrilateral circumscribed around a circle (a *circum quadrilateral*). Recall that the two tangents from a point outside a circle to the circle are equal in length.

Now, without using construction or measurement, work through the following questions using the diagram that shows a quadrilateral with all four sides tangent to a circle (*circum quadrilateral*).

1. Consider vertex A of circum quadrilateral $ABCD$. What can you say about the distances AP and AS ?
2. What can you say about the distances BP and BQ , the distances CQ and CR , and the distances DR and DS ?
3. Label AP as a , BP as b , CR as c , and DR as d , and write an expression for $AB + CD$.



4. From your observations in Questions 1 and 2, write an expression in terms of a , b , c , and d for $BC + AD$.
5. Compare Question 3 with Question 4. What do you notice?
6. Formulate your conclusion in Question 5 in your own words and discuss it with your partner or group.
7. From Question 6, what type of quadrilateral would $ABCD$ be if $AB = AD$?

CHECK BY CONSTRUCTION



Open the sketch **Circum Quad.gsp** and make some measurements and calculations to confirm your conclusion from Question 6. You'll find that you can drag points P , Q , R , and S , but not points A , B , C , and D .

Further Exploration

1. Construct the angle bisectors of all the angles of the circum quadrilateral. What do you notice? Can you explain your observation?
2. Which quadrilaterals (for example, parallelograms, rectangles, squares, kites, rhombuses, or squares) are special cases of a circum quadrilateral? Investigate by trying to drag your circum quadrilateral into the shapes of these special cases.
3. Is it possible to obtain a *concave* circum quadrilateral? If so, does your conjecture in Question 6 above still hold?

11. Since EF and HG are equal to $\frac{1}{2}AC$, and EH and FG are equal to $\frac{1}{2}BD$, $AC = BD$ implies that all the sides of $EFGH$ are equal; that is, it is a rhombus.
12. If $\overline{AC} \perp \overline{BD}$, the two pairs of opposite sides of $EFGH$ are perpendicular to each other. So all the angles are equal, which means that the rhombus becomes a square.

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13. No, we used only the property that it has equal diagonals. No other property of an isosceles trapezoid was used. Therefore, $ABCD$ need not be an isosceles trapezoid for the midpoint quadrilateral $EFGH$ to be a rhombus. The result would be true for any quadrilateral with equal diagonals.
14. Any quadrilateral with equal diagonals.
15. Carefully reflecting on logical explanations.

CHALLENGE A quadrilateral with equal diagonals can be constructed by first constructing a line segment, and then two circles with equal diameters. The endpoints of the diameters are the vertices of the quadrilateral. This construction can be used to drag the quadrilateral into convex, concave, and crossed forms.

Duality

It might be useful to point out to students the *angle-side* duality demonstrated by the two activities Kite Midpoints and Isosceles Trapezoid Midpoints. For example, the rectangle and rhombus are duals—the rectangle's angles are all equal and the rhombus's sides are all equal. The kite and isosceles trapezoid are also each other's duals, as illustrated in the following table:

Isosceles trapezoid	Kite
Two pairs of equal adjacent <i>angles</i>	Two pairs of equal adjacent <i>sides</i>
One pair of equal opposite <i>sides</i>	One pair of equal opposite <i>angles</i>
Circumscribed circle (<i>cyclic</i>)	Inscribed circle (<i>circum quad</i>)
An axis of symmetry through one pair of opposite <i>sides</i>	An axis of symmetry through one pair of opposite <i>angles</i>

LOGICAL DISCOVERY: CIRCUM QUAD (PAGE 68)

The purpose of this worksheet is to show students that new results in mathematics are sometimes discovered logically rather than always first by construction and measurement.

Prerequisites: Tangents to a circle are equal.

Sketches: **Circum Quad.gsp**. Additional sketches are **Concave Circ Quad.gsp**, **Circum Quad Converse.gsp**, and **Circum Hexagon.gsp**.

DISCOVERING

1. The distances are equal: $AP = AS$.
2. Similarly, $BP = BQ$, $CQ = CR$, and $DR = DS$.
3. $AB + CD = a + b + c + d$.
4. $BC + AD = b + c + a + d = a + b + c + d$.
5. $AB + CD = BC + AD$.
6. The sums of the two pairs of opposite sides of a circum quadrilateral are equal.
7. $ABCD$ would be a kite.

CHECK BY CONSTRUCTION

Encourage your students to check whether the result is also true for concave and crossed quadrilaterals (they may find this rather surprising) and to verify that these cases can be explained in a similar way.

Apart from stressing the value and power of logical reasoning in predicting in advance the outcomes of practical construction and measurement, such empirical/experimental testing gives concrete meaning to the results. Such testing is also often valuable in that it can provide us with counterexamples for certain special cases, which might necessitate a reformulation of the result or of its logical explanation (proof).

Further Exploration

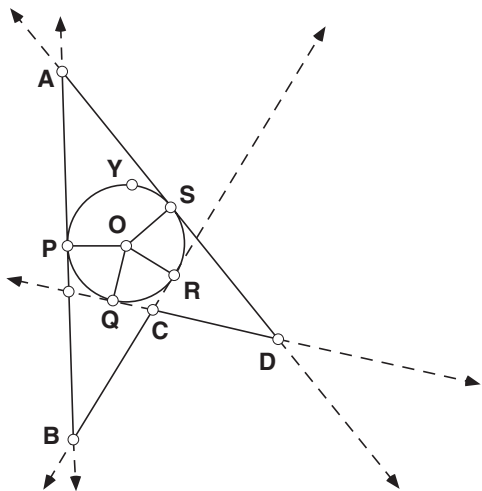
1. The angle bisectors are concurrent at the center of the incircle. The explanation is simple: The incenter is equidistant from all four sides (the radii of the incircle are perpendicular to the quadrilateral's sides). But

each angle bisector is the locus of all points equidistant from its two adjacent sides. Therefore, each angle bisector must pass through the incenter.

Conversely, this is a necessary condition for any polygon to be circumscribed around a circle. (For example, for a polygon to have an incircle, it must have a point that is equidistant from all the sides. Therefore, the angle bisectors must be concurrent.)

- It's possible to drag to obtain special cases such as a (general) kite, a (general) rhombus, or a square.
- By constructing the intersection of the tangential lines as in the first figure shown, we can obtain a concave circum quad in which the extensions of two sides are now tangent to the incircle. Suppose we label the tangent segments as before; then $AB + CD = (a + b) + (d - c)$ and $BC + AD = (b - c) + (a + d)$, which are clearly equal. A sample sketch is given in **Concave Circ Quad.gsp**.

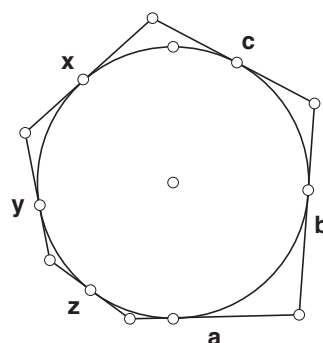
DA = 14.363 cm
BC = 4.092 cm
DA + BC = 18.46 cm
DC = 6.910 cm
AB = 11.545 cm
DC + AB = 18.46 cm



You may wish to encourage even further exploration by asking your students how they would try to generalize the result to other polygons.

Both the convex and concave cases can also be generalized to certain types of $2n$ -gons ($n > 1$) in which the two sums of *alternate* sides are equal. A sample sketch is shown in the second figure and is given in **Circum Hexagon.gsp**.

$a = 4.504$ cm
 $c = 4.197$ cm
 $y = 4.250$ cm
 $a + c + y = 12.951$ cm
 $x = 4.173$ cm
 $z = 4.957$ cm
 $b = 3.822$ cm
 $x + z + b = 12.951$ cm



The following theorem in this regard is proved in de Villiers (1993; 1996, 183–187):

If $A_1A_2 \dots A_{2n}$ ($n > 1$) is any circumscribed $2n$ -gon in which vertex $A_i \rightarrow A_{i+k}$ (vertex A_i is joined to A_{i+k}), the two sums of alternate sides are equal (where k is the total turning we would undergo by walking around the perimeter of the polygon).

You may want to encourage your students to investigate whether the converse of the circum quadrilateral result for the sides is also true. That is, is a quadrilateral with the two sums of opposite sides equal necessarily a circum quadrilateral? A sketch called **Circum Quad Converse.gsp** is provided to assist such an investigation. To prove the converse, you can use proof by contradiction (see de Villiers 1996). Note that the converse is true only for quadrilaterals; it is not, in general, true for $2n$ -gons.

Duality

The *angle-side* duality referred to in the Teacher Notes for the Isosceles Trapezoid Midpoints activity is also neatly displayed between the circum and cyclic quadrilaterals. For example, in the circum quadrilateral, the two sums of opposite sides are equal, whereas in the cyclic quadrilateral, the two sums of opposite angles are equal. The square and the parallelogram are self-dual in relation to these two concepts, as shown in the following tables.

Square	
Circumscribed circle (cyclic)	Inscribed circle (circum quad)
An axis of symmetry through each pair of opposite sides	An axis of symmetry through each pair of opposite angles

Parallelogram	
Equal opposite angles	Equal opposite sides
No circumscribed circle	No inscribed circle

Using this angle-side duality, the quadrilaterals can be classified as shown. Note that the vertical line of symmetry can be used to find the dual of any particular quad by reflection in it. For a more detailed discussion of this duality, consult de Villiers (1996).

