Name(s):

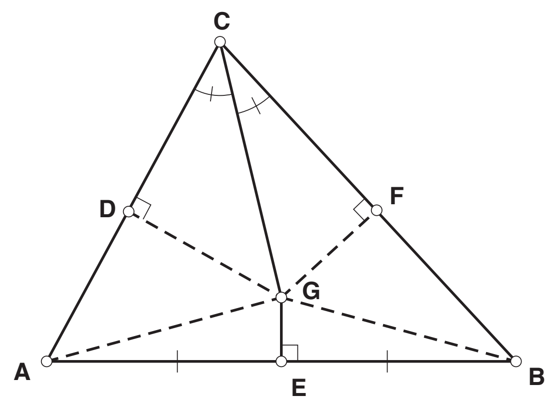
Logical Paradox

Sometimes a seemingly correct logical argument can lead to a paradox. Work through the following logical argument in relation to the diagram shown. Do not use Sketchpad yet; you will use it later to check the validity of this argument.

CONJECTURE

The diagram on the right shows the following construction.

- Triangle *ABC* is any arbitrary triangle.
- $\overline{CG}$ is on the angle bisector of angle *ACB*, and $\overline{GE}$ is the perpendicular bisector of $\overline{AB}$.
- $\overline{GD}$ is perpendicular to $\overline{AC}$, and $\overline{GF}$ is perpendicular to $\overline{BC}$.



1. What can you say about triangles *CGD* and *CGF*? Why?

2. From Question 1, what can you conclude about *DG* and *FG*?

3. What can you say about *AG* and *BG*? Why?

4. What can you now conclude about triangles *GDA* and *GFB*? Why?

5. From Question 4, what can you conclude about *DA* and *FB*?

6. From Question 1, what can you conclude about *CD* and *CF*?

7. What can you now conclude about $CD + DA$ and $CF + FB$, and therefore about *CA* and *CB*?

8. From Question 7, what type of triangle is *ABC*?

REFLECT

Is this argument valid for *any* triangle *ABC*? What is the problem? Where is the mistake? Discuss with your partner or your group.

CHECK BY CONSTRUCTION

Make an accurate construction in Sketchpad to check the sketch that provides the basis of the logical argument. What do you notice? What important lesson can you learn from this?

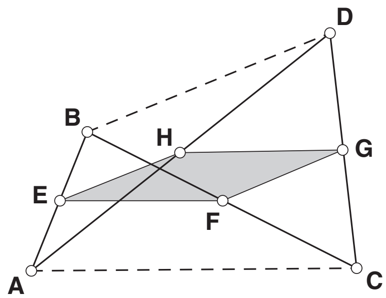
To construct an angle bisector, select three points on the angle, making sure the vertex is your middle selection. Then choose **Angle Bisector** from the Construct menu.

To construct a perpendicular, select a point and a straight object. Then choose **Perpendicular Line** from the Construct menu.

Applying the above formula and definition of area in a crossed quadrilateral (see figure), we find that diagonal AC decomposes its area as follows:

$$(ABCD) = (ABC) + (CDA) = (ABC) - (ADC)$$

In other words, this formula forces us to regard the "area" of a crossed quadrilateral as the difference between the areas of the two small triangles ABO and ODC. (Note that diagonal BD similarly decomposes $(ABCD)$ into $(BCD) + (DAB) = -(DCB) + (DAB)$). An interesting consequence of this is that a crossed quadrilateral will have zero "area" if the areas of triangles ABO and ODC are equal.



Using this valuable notation, the result can now simultaneously be proved for all three cases (convex, concave, and crossed) as follows:

$$(EFGH) = (ABCD) - (AEH) - (FCG) - (EBF) - (DHG)$$
$$= (ABCD) - \frac{1}{4}(ABD) - \frac{1}{4}(CDB) - \frac{1}{4}(BCA) - \frac{1}{4}(DAC)$$
$$= (ABCD) - \frac{1}{4}(ABCD) - \frac{1}{4}(ABCD)$$
$$= \frac{1}{2}(ABCD)$$

LOGICAL PARADOX (PAGE 80)

This worksheet is based on an example that has often been used (wrongly) to try to motivate a need for proof among students. Basically, students are told that this example illustrates that diagrams may be deceiving and therefore unreliable. Consequently, reliance only on experimental evidence is unreliable and we thus require formal proof.

However, this example actually illustrates the importance of making (reasonably) *accurate* diagrams when constructing proofs, rather than showing that diagrams are unreliable. In fact, the false conclusion that all triangles are isosceles shows how easily a correct logical argument can lead to a fallacy because of a construction error, or a mistaken assumption, in a sketch. Instead of motivating a need for proof, such examples actually emphasize the importance of experimental testing (i.e., the *accurate* construction of some examples), noting with care the relative positions of points, lines, and so on that are essential to the proof. Although a French mathematician once said "Geometry is the art of drawing correct conclusions from incorrectly drawn sketches," this example dramatically shows that they should not be constructed too incorrectly!

Prerequisites: Knowledge of conditions for congruency.

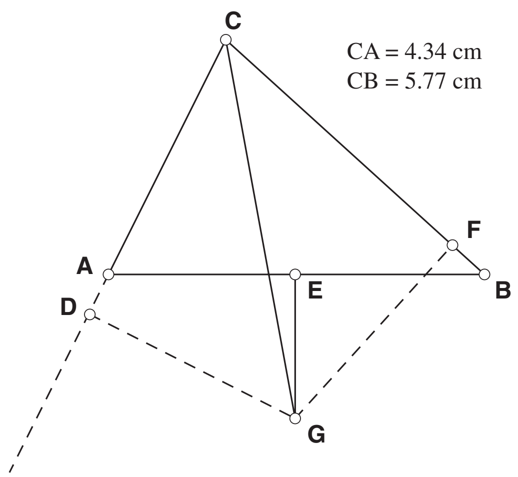
Sketch: Paradox.gsp (This sketch should be given to students only at the end of the worksheet, after they have worked through the logical argument based on the faulty diagram.)

CONJECTURE

1. Triangles CGD and CGF are congruent (SAA).
2. $DG = FG$.
3. $AG = BG$, since G lies on the perpendicular bisector of $\overline{AB}$.
4. Triangles GDA and GFB are congruent (90°, S, S).
5. $DA = FB$.
6. $CD = CF$.
7. $CD + DA = CA = CF + FB = CB$.
8. Therefore, ABC is isosceles.

REFLECT/CHECK

Although the argument itself is quite correct, the problem arises from an incorrectly drawn sketch. For example, when we actually construct this diagram in Sketchpad (or with paper and pencil), the point *G* always falls outside, and one of the points *D* or *F* always falls *inside* the triangle while the other falls *outside,* and therefore invalidates the "proof," as the diagram below demonstrates. (Note that if it is given that $CA = CB$, the angle bisector of angle *C* and the perpendicular bisector of *AB* coincide and there is no unique point *G*.)



The experimental observation that *G* always falls outside and that *one* of the points *D* or *F* always falls *inside* the general triangle while the other falls *outside* is proved in Movshovitz-Hadar and Webb (1998, 74–75). It is also possible to construct a simpler argument based on symmetry, starting with the assumption that *ACB* is isosceles, with $AC = BC$, and then considering what happens if C is moved to the left or the right of the perpendicular bisector of $\overline{AB}$.

CYCLIC QUADRILATERAL CONVERSE (PAGE 82)

Following the Areas activity, in which students are introduced to a false result, this worksheet focuses on further elaborating the verification function of a proof. In other words, it can be used to convince or to remove lingering doubts. However, it is important that you not yet present proof at this stage to your students as the *only* accepted means of verification in mathematics. Instead it should be emphasized as an *additional* or *complementary* path to verification/conviction.

Prerequisites: Cyclic Quadrilateral activity. Knowledge of exterior angle theorem for a triangle.

Sketch: Cyclic Quad.gsp. An additional sketch is **Cyclic Quad 2.gsp.** (In this dynamic sketch, a quadrilateral *EFGH* has been constructed with opposite angles *HEF* and *HGF* supplementary by constructing them respectively equal to angles *DCA* and *DCB* lying adjacent to each other on a straight line. The sketch shows that the perpendicular bisectors are always concurrent and that a circumcircle always passes through all four vertices.)

Answers to Introductory Questions

Its opposite angles are supplementary.

The converse: If the opposite angles of a convex quadrilateral are supplementary, the quadrilateral is cyclic.

CONJECTURE

1. Quadrilateral *ABCD* is cyclic.
2. The sketch appears to support the formulated converse.
3. Responses may vary.

CHALLENGE Responses may vary, but it is anticipated that not all students will be entirely convinced that the conjecture is always true and that this will create a need for additional verification (that is, a logical proof).