Let $a, b, c, d, e, f, g, h, \ell, a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3 \in \mathbb{C}$ the affixes of the points $A, B, C, D, E, F, G, H, I, A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3$.

ΔABG equilateral $\Leftrightarrow r_{B,60^\circ}(A) = G \Leftrightarrow$

$g = b + (a-b)(\cos 60^\circ + i\sin 60^\circ) \Leftrightarrow$

$g \overset{①}{=} -\varepsilon a - \varepsilon^2 b$; where $\varepsilon = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$; $\varepsilon^2+\varepsilon+1=0$, $\varepsilon^3 = 1$.

ΔDHC equilateral $\Leftrightarrow r_{D,60^\circ}(C) = H \Leftrightarrow$

$h \overset{②}{=} -\varepsilon c - \varepsilon^2 d$.

ΔIEF equilateral $\Leftrightarrow r_{F,60^\circ}(E) = I \Leftrightarrow$

$\ell \overset{③}{=} -\varepsilon e - \varepsilon^2 f$.

A_1 centroid of triangle $FGC \Rightarrow$

$a_1 = \frac{c+f+g}{3} \overset{①}{=} \frac{c+f-\varepsilon a - \varepsilon^2 b}{3}$ ④

B_1 centroid of triangle $BEH \Rightarrow$

$b_1 = \frac{b+e+h}{3} \overset{②}{=} \frac{b+e-\varepsilon c - \varepsilon^2 d}{3}$ ⑤

C_1 centroid of triangle $DIA \Rightarrow$

$c_1 = \frac{a+d+\ell}{3} \overset{③}{=} \frac{a+d-\varepsilon e - \varepsilon^2 f}{3}$ ⑥

$$-\varepsilon b_1-\varepsilon^2 a_1 \overset{④;⑤}{=\!=} \frac{1}{3}\left(-\varepsilon(b+e-\varepsilon c-\varepsilon^2 d)-\right.$$
$$\left.-\varepsilon^2(c+f-\varepsilon a-\varepsilon^2 b)\right)=\frac{1}{3}\left(\underbrace{\varepsilon^3}_{1}a+(\underbrace{\varepsilon^4-\varepsilon}_{0})b+\right.$$
$$\left.+(\underbrace{\varepsilon^2-\varepsilon^2}_{0})c+\underbrace{\varepsilon^3}_{1}d-\varepsilon e-\varepsilon^2 f\right)=\frac{1}{3}(a+d-\varepsilon e-\varepsilon^2 f)=$$
$$\overset{⑥}{=} c_1 \Rightarrow c_1=-\varepsilon b_1-\varepsilon^2 a_1 \Leftrightarrow \Delta A_1B_1C_1 \text{ is equilateral.}$$

A_2 centroid of triangle DGE $\Rightarrow$

$$a_2=\frac{d+e+g}{3}\overset{①}{=}\frac{d+e-\varepsilon a-\varepsilon^2 b}{3} \quad ⑦$$

B_2 centroid of triangle AHF $\Rightarrow$

$$b_2=\frac{a+f+h}{3}\overset{②}{=}\frac{a+f-\varepsilon c-\varepsilon^2 d}{3} \quad ⑧$$

C_2 centroid of triangle BIC $\Rightarrow$

$$c_2=\frac{b+c+l}{3}\overset{③}{=}\frac{b+c-\varepsilon e-\varepsilon^2 f}{3} \quad ⑨$$

$$-\varepsilon c_2-\varepsilon^2 a_2 \overset{⑦;⑨}{=\!=} \frac{1}{3}\left(-\varepsilon(b+c-\varepsilon e-\varepsilon^2 f)-\right.$$
$$\left.-\varepsilon^2(d+e-\varepsilon a-\varepsilon^2 b)\right)=\frac{1}{3}\left(\underbrace{\varepsilon^3}_{1}a+(\underbrace{\varepsilon^4-\varepsilon}_{0})b-\varepsilon c-\right.$$
$$\left.-\varepsilon^2 d+(\underbrace{\varepsilon^2-\varepsilon^2}_{0})e+\underbrace{\varepsilon^3}_{1}f\right)=\frac{1}{3}(a+f-\varepsilon \varepsilon-\varepsilon^2 d)=$$
$$\overset{⑧}{=} b_2 \Rightarrow b_2=-\varepsilon c_2-\varepsilon^2 a_2 \Leftrightarrow \Delta A_2B_2C_2 \text{ is equilateral.}$$

A_3 centroid of triangle IGH $\Rightarrow$

$$a_3 = \frac{g+h+l}{3} \overset{(1),(2),(3)}{=} \frac{1}{3}\left(-\varepsilon a - \varepsilon^2 b - \varepsilon c - \varepsilon^2 d - \varepsilon e - \varepsilon^2 f\right) \quad (10)$$

B_3 centroid of triangle ACE $\Rightarrow$

$$\Rightarrow b_3 = \frac{a+c+e}{3} \quad (11)$$

C_3 centroid of triangle BDF $\Rightarrow$

$$\Rightarrow c_3 = \frac{b+d+f}{3} \quad (12)$$

Then:

$$-\varepsilon c_3 - \varepsilon^2 a_3 = \frac{1}{3}\left(-\varepsilon(b+d+f) - \varepsilon^2\left(-\varepsilon a - \varepsilon^2 b - \varepsilon c - \varepsilon^2 d - \varepsilon e - \varepsilon^2 f\right)\right) = \frac{1}{3}\left(\underbrace{\varepsilon^3}_{1} a + \underbrace{(\varepsilon^4 - \varepsilon)}_{0} b + \underbrace{\varepsilon^3}_{1} c + \underbrace{(\varepsilon^4 - \varepsilon)}_{0} d + \underbrace{\varepsilon^3}_{1} e + \underbrace{(\varepsilon^4 - \varepsilon)}_{0} f\right) =$$

$$= \frac{1}{3}(a+c+e) \overset{(11)}{=} b_3 \Rightarrow$$

$$\Rightarrow b_3 = -\varepsilon c_3 - \varepsilon^2 a_3 \Rightarrow \Delta A_3 B_3 C_3 \text{ is equilateral}$$

q.e.d.

Dao Than Oai's generalization of Napoleon's theorem

Michael de Villers.

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