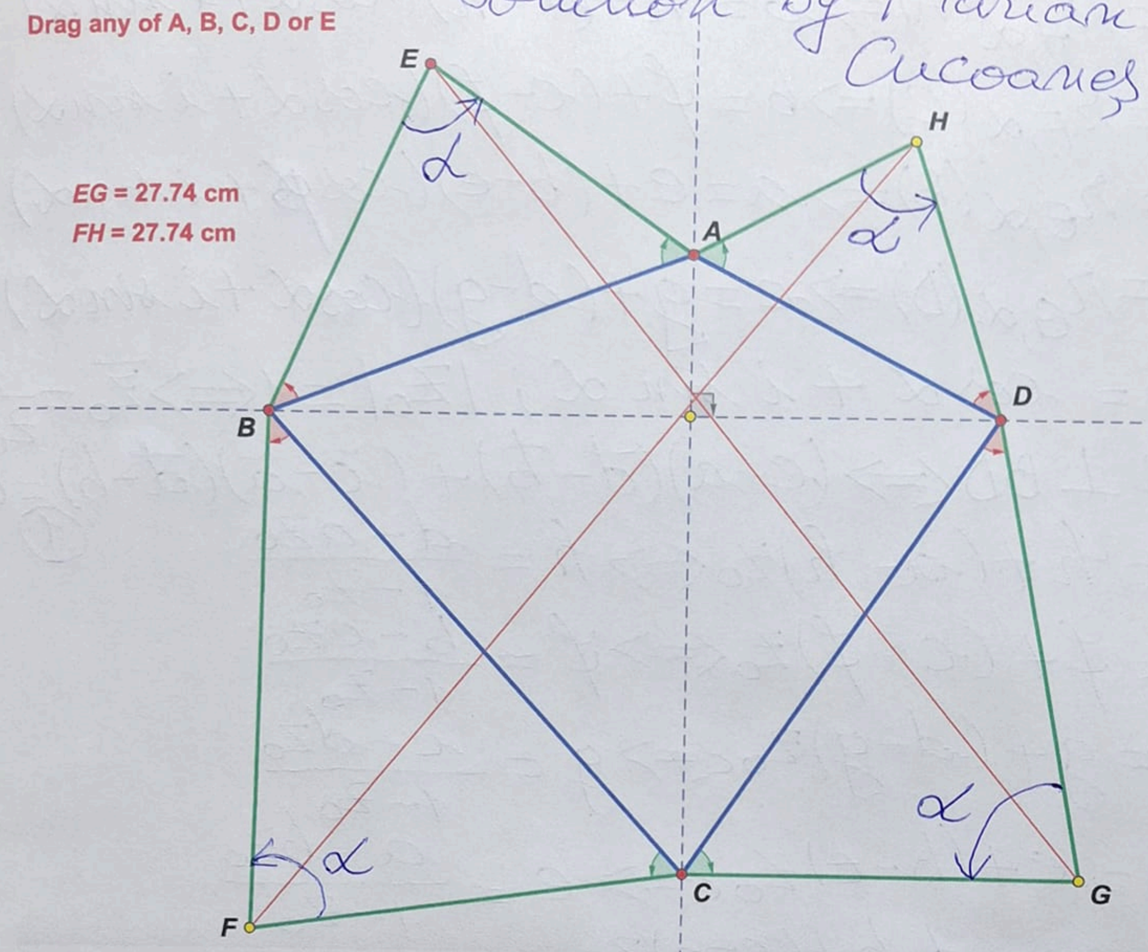
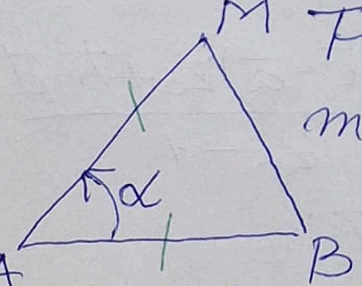
Solution by Marian Cucoanes



$\measuredangle BEA = \measuredangle AHD = \measuredangle DGC = \measuredangle CFB = \alpha.$

<u>Lemma</u> X point and $x \in \mathbb{C}$ the complex coordinate of point X.



Then: $r_{A,\alpha}(B) = M \iff$

$$m = a + (b-a)(\cos\alpha + i\sin\alpha).$$

Denote by: a, b, c, d, e, f, g, h the complex coordinates of the points. A, B, C, D, E, F, G, H

$D = r_{H,\alpha}(A) \Rightarrow d = h + (a-h)(\cos\alpha + i\sin\alpha)$

$B = r_{F,\alpha}(C) \Rightarrow b = f + (c-f)(\cos\alpha + i\sin\alpha)$

$A = r_{E,\alpha}(B) \Rightarrow a = e + (b-e)(\cos\alpha + i\sin\alpha)$

$C = r_{G,\alpha}(D) \Rightarrow c = g + (d-g)(\cos\alpha + i\sin\alpha)$

$z_0 = \cos\alpha + i\sin\alpha\ ;\ |z_0| = 1 \Leftrightarrow \bar{z}_0 = \frac{1}{z_0}$

$AC \perp BD \Leftrightarrow (c-a)(\bar{d}-\bar{b}) + (\bar{c}-\bar{a})(d-b) = 0$. (1)

$d = h + (a-h)z_0 \Leftrightarrow h = \frac{d - az_0}{1-z_0}$

$b = f + (c-f)z_0 \Leftrightarrow f = \frac{b - cz_0}{1-z_0}$

$c = g + (d-g)z_0 \Leftrightarrow g = \frac{c - dz_0}{1-z_0}$

$a = e + (b-e)z_0 \Leftrightarrow e = \frac{a - bz_0}{1-z_0}$

$h - f = \frac{d - b + (c-a)z_0}{1-z_0}$

$\bar{h} - \bar{f} = \frac{\bar{d} - \bar{b} + (\bar{c}-\bar{a})\bar{z}_0}{1-\bar{z}_0} = \frac{(\bar{d}-\bar{b})z_0 + \bar{c} - \bar{a}}{z_0 - 1}$

$g - e = \frac{c - a + (b-d)z_0}{1-z_0}$.

$\bar{g} - \bar{e} = \frac{\bar{c} - \bar{a} + (\bar{b}-\bar{d})\bar{z}_0}{1-\bar{z}_0} = \frac{(\bar{c}-\bar{a})z_0 + \bar{b} - \bar{d}}{z_0 - 1}$.

$HF^2 = |h-f|^2 = (h-f)(\bar{h}-\bar{f}) =$

$= \left[\frac{d - b + (c-a)z_0}{1-z_0}\right] \cdot \left[\frac{(\bar{d}-\bar{b})z_0 + \bar{c} - \bar{a}}{z_0 - 1}\right] \Rightarrow$

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$$HF^2 = \frac{|d-b|^2 z_0 + |c-a|^2 z_0 + (c-a)(\bar{d}-\bar{b})z_0^2 + (d-b)(\bar{c}-\bar{a})}{-(z_0-1)^2} \quad (2)$$

$$GE^2 = |g-e|^2 = (g-e)(\bar{g}-\bar{e}) =$$

$$= \left[\frac{c-a+(b-d)z_0}{1-z_0}\right] \cdot \left[\frac{(\bar{c}-\bar{a})z_0 + \bar{b}-\bar{d}}{z_0-1}\right] \Rightarrow$$

$$GE^2 = \frac{|b-d|^2 z_0 + |c-a|^2 z_0 + (b-d)(\bar{c}-\bar{a})z_0^2 + (c-a)(\bar{b}-\bar{d})}{-(z_0-1)^2} \quad (3)$$

(1) and (2) and (3) $\Rightarrow HF^2 = GE^2 \Rightarrow$

$\Rightarrow HF = GE$ q.e.d