Why does it Work? A Mathematical Explanation and Further Generalization of a Card Trick

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INTRODUCTION

I was recently shown an interesting card trick by a young family relation who then specifically asked me if I could figure out why it worked. Of significance is that both she and her father (who originally showed her the trick) were *entirely convinced* of its validity. They had no doubt that it would *always* work, it was simply that they couldn't figure out *why* it worked. This highlights the important, fundamental distinction I myself, and others, have consistently made between *conviction* and *explanation* in mathematics (De Villiers, 1990; Hanna, 1989; Harel, 2013). Unfortunately many curricular activities at school, and most textbooks, do not exploit this distinction to present proof as a meaningful experience to learners. Most often, doing a mathematical proof is only motivated as a means of gaining conviction, while its potential explanatory power is all but neglected.

THE CARD TRICK

The trick works as follows. Start with a standard pack of 52 cards. Jacks, Queens and Kings are all assigned the value of 10, and all other cards are assigned their respective face values. Begin dealing out the cards into piles in the following fashion: starting with the value of the first card, pack out sufficient additional cards to get a total of 13. (For example, if the first card is a 9, then pack out four additional cards onto the pile while counting 10, 11, 12, 13, thus giving a total of 5 cards in the pile). Continue dealing out the rest of the cards into piles in the same fashion until you can't continue any more, i.e. stop when you are left with insufficient cards to form a pile in this fashion. Now select any three of the piles. Put them on one side and count the number of cards in the other piles. To that total add the number of cards that were left over after the initial dealing into piles, and subtract 10 – and *voila!* The answer obtained equals the sum of the values of the initial card in each of the three piles that were put on one side. Magic!

To better understand the card trick, let us illustrate it with an example. Suppose the first card is a 9, then the first pile will have five cards. If the first card of the next pile is say, 5, then the second pile will have nine cards in it. Let's assume the next first cards in order for each pile are say 1, 2 and 3 so that each of these piles will have 13, 12 and 11 cards respectively in total. Since the total number of cards is now 5 + 9 + 13 + 12 + 11 = 50, we are left with two cards, and no further piles can be formed. Suppose we choose the first 3 piles, with respective first cards 9, 5 and 1, to put on one side. We then add the number of cards in the other piles to the number of cards left over and subtract 10 to get: 12 + 11 + 2 – 10 = 15. But that is the same as the sum of the first cards of the three chosen piles: 9 + 5 + 1 = 15. *Voila!*

Before reading on, take a pack of cards and test the trick to see if it works by doing a few examples. If you are happy that it seems to work, try to figure out *why* it will *always* work.

THE EXPLANATION

The trick can easily be explained using elementary algebra. If we let the value of the first card in each dealt out pile be a, b, c, d, etc. then the number of cards in each pile will be $14 - a$, $14 - b$, $14 - c$, $14 - d$, etc.

The number of remaining cards, i.e. those not dealt into piles is thus:

$$r = 52 - [(14 - a) + (14 - b) + (14 - c) + (14 - d) \ldots]$$

Suppose we chose the first three piles to put on one side. Rearranging the above equation:

$$\begin{aligned} r + [(14 - d) + (14 - e) \ldots] &= 52 - [(14 - a) + (14 - b) + (14 - c)] \\ &= 52 - 14 \times 3 + [a + b + c] \\ &= 10 + [a + b + c] \end{aligned}$$

This shows that $r + [(14 - d) + (14 - e) \ldots] - 10 = a + b + c$, i.e. that the remaining cards, r, plus those not in the three chosen piles minus 10 will always equal the sum of the values of the initial card in each of the three chosen piles, i.e. a, b and c. The same situation will clearly apply to any combination of chosen piles, and this completes the explanation (and proof).

FURTHER REFLECTION

It is often the case that further reflection on a proof in the style of Polya (1945) may reveal useful generalizations or specializations, what has been termed the 'discovery' function of proof (De Villiers, 1990). Learners in class can easily be led by appropriate questions and pointers to the following conclusions, nicely illustrating this discovery function of proof.

- From the explanatory proof one can immediately see that one need not number the picture cards with the value 10, but that it would also work in exactly the same way if the Jack, Queen and King were assigned, for example, the values of 11, 12 and 13 respectively.
- In addition, one can also see that the 'end' or 'target' number for the cards in each pile need not be 13, but could be any number. For example, if one dealt out each pile up to 14 as the target number, and chose three piles as before, one would need to deduct $52 - 15 \times 3 = 7$ from the remainder.
- Lastly, the number of piles chosen doesn't have to be three either. For example, one could choose two piles to put on one side, in which case (for the original trick) instead of subtracting 10 one would need to deduct $52 - 14 \times 2 = 24$. (Four piles for the original trick will only work if you add 4 to the remainder, since $52 - 14 \times 4 = -4$. Of course, for a smaller total in each pile one could choose more piles if one wanted to deduct at the end.

CONCLUDING COMMENTS

This card trick (or similar self-working card tricks) can easily be used in the classroom when proof is introduced for the first time to secondary school learners (as well as in prospective mathematics teacher education courses). Apart from hopefully amazing and puzzling learners, it not only illustrates the explanatory power of a deductive proof, but also its discovery function by revealing different ways in which the trick can be generalized.

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