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Mathematical Applications, Modeling and Technology

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Introduction

The new South African mathematics curriculum through all the grades strongly emphasizes a more relevant, realistic approach focusing a lot on applications of mathematics to the real world. This is in line with curriculum development in most other countries. For example, the influential *NCTM Standards* sums it up succinctly in the following respective standards for instructional programs in algebra and geometry from prekindergarten through grade 12 (respectively from http://standards.nctm.org/document/chapter3/alg.htm http://standards.nctm.org/document/chapter3/geom.htm):

(*Students should be enabled to in …*)

Algebra

- *use mathematical models to represent and understand quantitative relationships;*

Geometry

- *use visualization, spatial reasoning, and geometric modeling to solve problems.*

Like the *NCTM Standards*, the new South African curriculum now also encourages for the first time the use of appropriate technology in modeling and solving real-world problems as follows (Dept. of Education, 2002, p. 3):

- *use available technology in calculations and in the development of models.*

But how do our learners and students interpret mathematics and its relationship to the real world? What is their proficiency in applying mathematics, modeling real-life problems (and using technology efficiently)?

Two illustrative examples

I've often given the following problem as one of a range of problems to prospective primary mathematics teachers to give to their learners during practice teaching:

Sally has some dolls and is given 5 more dolls by her grandmother so that she now has 12 dolls in total. How many dolls did she have in the beginning?

Much to these prospective teachers' surprise, a substantial number of learners come up with the seemingly meaningless answer of 17 by simply adding 12 and 5. Firstly, note that the problem here is clearly NOT one of computational ability, as they CORRECTLY added 12 and 5. Furthermore, this kind of response was sometimes more prevalent among older children in Grade 3 than among younger ones in Grades 1 or 2. How do we explain this?

Without getting in to deeply analyzing the reasons for this error, it would rather seem that it is a case of interpreting or modeling the situation incorrectly. Moreover, fuelled perhaps by past experiences of school mathematics being largely divorced from the real world, learners clearly fail to test or evaluate the meaningfulness of their answers in the given context. Further research also indicates that some of these learners just picked out the two numbers and added them because of the word 'more' – a dubious strategy sometimes actually taught by textbooks and teachers for 'simplifying' word problems by identifying certain key words that supposedly suggests the type of arithmetic operation to be used!

Here is another problem from the 1984 RUMEUS (Research Unit for Mathematics Education at the University of Stellenbosch) Algebra test I've often given to our prospective high school teachers for use during practice teaching and was also part of mini investigation of 40 Grade 11 learners by Kowlesar (1992):

A sightseeing trip around the Durban beachfront is given by the following formulae:

Taxi A: $C = 80 + 25t$

Taxi B: $C = 100 + 20t$

Draw a graph to show ho w the cost C (in cents) varies with the time t for Taxi A and Taxi B on the same graph.

Which of the two taxis is cheaper?

Shockingly in Kowlesar (1992), it was found that NONE of the learners could draw the correct graphs or determine when one was cheaper than the other! The most popular response was to plot two "*coordinates*" namely, (80, 25*t*) and (100, 20*t*) on a cost-time graph. The learners' explanations were illuminating of some of their underlying misconceptions, for example:

I thought that 80 was the distance and 25t the time, therefore I plotted these as coordinates.

Did not know what to do. Can't make certain of what was wanted. Is it cost against time? How do you plot 25t and 20t?

Dealing with money and time. I won't be able to plot points like 100t.

I did not have any idea what to do. I've never seen a function in that form before.

So what was the problem? Was it because they hadn't yet been taught linear functions and graphs or maybe just not remembered the work they'd done earlier on?

However, as became apparent through interviews with some of these Grade 11 learners, they were ALL able to easily draw the graphs of $y = 80 + 25x$ and $y = 100 + 20x$. The problem therefore clearly was not one of instrumental proficiency with linear functions and graphs (in standard form with variables x and y), but one of recognizing and handling linear functions and variables in REAL WORLD CONTEXTS.

Decontextualised teaching and learning

The above two examples clearly show that knowing and being able to carry out certain mathematical algorithms and procedures provides NO definite guarantee that learners are able to interpret and solve real world problems meaningfully. Proposals by mathematics educators and professional associations over the past few decades that what needs to be done is to more regularly use real world contexts as STARTING POINTS for developing the mathematics to be learnt, has still to a large degree, fallen on deaf ears and is sometimes even met with fierce resistance by traditionalists.

However, the root of the problem seems quite clear. Is it any wonder that children cannot translate from "pure" mathematics to practical situations, when mathematics is mostly taught in a DECONTEXTUALISED fashion; that is, in a "vacuum", completely devoid of any relationship to the real world?

For example, in the first problem, just knowing how to add or subtract provides no guarantee that either of these operations will be recognized as the appropriate model for a specific practical situation. Similarly, if linear functions as in the second problem are decontextualised and just taught instrumentally, the chances are that learners would later be unable to apply them to real world situations.

Indeed, one could argue, with substantial support from research evidence, that if mathematical skills and content have been taught completely divorced from any real world interpretations and meanings, attempts to do so at a later stage are mostly futile.

Some years ago as described in De Villiers (1992), I was observing a student teacher dealing with long division in Grade 5. After some preliminary discussion and demonstration, he gave the problem 8.048 kg ÷ 4 to the class, upon which they all easily and efficiently solved the problem correctly. A very successful lesson it would certainly seem, not so?

I then asked the class to write a little story describing a real problem situation involving decimals to which this calculation would produce the answer. The results were shocking!

Of the class of 31, not a single child described a relevant, meaningful context, with 30 writing stories not even involving mass at all, for example:

There are 8.048 cows. We must divide them equally among 4 people. How much does each one get?

A farmer has 8048 oranges which he wants to divide into 4 equal groups for needy people.

The only child who described a situation involving mass, wrote the following:

If John bought a tape recorder of mass 8.048 kg. He has to divide it amongst four people. Each one gets 2.012 kg.

Although one's initial response may be one of laughter, further reflection is more likely to move one to tears. Is this what mathematics means to our children? Is this decontextualised, irrelevant and meaningless learning still going on in most of our schools? What is the use if children can do arithmetic operations, but they they do not even know what it means nor what it is useful for?

Modeling as a teaching strategy

In contrast to traditional decontextualised teaching, modeling of real world contexts can effectively be used as a teaching strategy, i.e. where the mathematical theory is initiated by, and directly developed from a practical situation (to which it can later also be re-applied). For example, in my book *Boolean Algebra at School*, the entire theory is systematically modeled from several electrical switching circuit problems, and only in the last section is it formally axiomatised, and are proofs developed (De Villiers, 1987).

It seems axiomatic that when mathematical concepts, algorithms, theorems, etc. are directly abstracted from practical contexts and problems, the links between such content and the real world must be stronger than when it is attempted to create such links only after the content has been presented in a vacuum. Groundbreaking work by especially the Freudenthal Institute in the Netherlands over several decades, as well as the Problem-Centred Approach of the University of Stellenbosch (Murray, Olivier & Human, 1998) since the mid 1980's, using precisely such a modeling approach, has shown not only its feasibility, but also substantial gains in children's ability to relate mathematics to the real world meaningfully (as well as increased conceptual understanding generally).

In traditional decontextualised teaching, children are often appeased when asking the teacher why they need to learn some theory or algorithm by usually being told that they will LATER ON see its practical or theoretical applications – which sometimes takes weeks or months – hardly motivating at all! In contrast, using modeling as a teaching strategy by starting with real world (or theoretical) applications has much higher motivational value as it immediately places the usefulness or value of the content in the foreground.

However, it should also be pointed out that not all mathematical topics in the school curriculum might be equally well suited for development through modeling as a teaching strategy. Historically, we should remember that the source of new mathematics, particularly from the 19th century onwards, was not always the real world, but motivated quite often by needs for symmetry and structure, to solve theoretical problems, or from further reflection, abstraction and generalization of existing mathematics.

Unrealistic problems and contexts

There exist numerous examples in which "reality" is thoroughly degraded in unrealistic, contrived and artificial problems. Consider for example, the following two examples:

A person saves 1c on the first day; 2c on the second day; 4c on the third day; etc. How much does the person save in a year?

Quite clearly no normal person would exhibit the strange saving habit in the first problem; in fact, no-one in the world, including Bill Gates, would have enough money to keep on saving in this manner for a year! The second problem may be amusing and interesting, but clearly it is a "*theoretical*" problem simply clothed in practical terms – it bears no relationship to any conceivable REAL WORD problem.

In the 2005 Grade 9 Common Task for Assessment for GET (General Education & Training), according to Van Etten & Adendorff (2006), one of the tasks for learners apparently was to predict the total number of elephants in Africa after a decade, only given a total number of 1.3 million elephants in 1979 and a poaching figure of 200 elephants on average per day slaughtered. Clearly, this is highly unrealistic as neither the average birth rate nor the average death rate due to natural disease, aging, etc. is given. More over, it is simplistically assuming that a linear model applies. If the purpose of the task was for learners to realize the inadequacy of the given data and use of a linear model, it can still be defended, but it's not clear whether that was the intention at all.

However, we should also be cautious not to lose genuine mathematics if too much focus is put on the real world context. Jan De Lange (1996: 5) humorously illustrates this with the following (exaggerated) example of the changes in mathematics education over past number of decades:

** in 1960: A woodcutter sells on load of wood for $100. If the production cost is four fifths of the selling price, what is his profit?*

** in 1970 (modern mathematics): A woodcutter sells a set of L of wooden blocks for a set M of dollars. The number of elements of M is 100. The subset C of the cost has four fifths of the number of elements of M. Draw C as a subset of M and determine the number of elements of the subset which gives the profit.*

** in 1980 (back to basics): A woodcutter sells one load of wood for $100. The profit is $20. Underline the number 2 and discuss its place value.*

** in 1990 (socio-politically directed education): By felling beautiful trees, the woodcutter earns $20. Classroom discussion: How would the birds and squirrels feel about the felling of the trees, etc.?*

The modeling process

The process of mathematical modeling essentially consists of three steps or stages as illustrated in Figure 1, namely:

1. construction of the mathematical model
2. solution of the model
3. interpretation and evaluation of the solution.

Stage 1

During the construction of the model one or more prerequisites are often necessary, for example:

- the making of appropriate assumptions to simplify the situation
- data collection, tabulation, graphical representation, data transformation, etc.
- identification and symbolization of variables
- the construction of suitable formulae and/or representations like scale drawings or working models

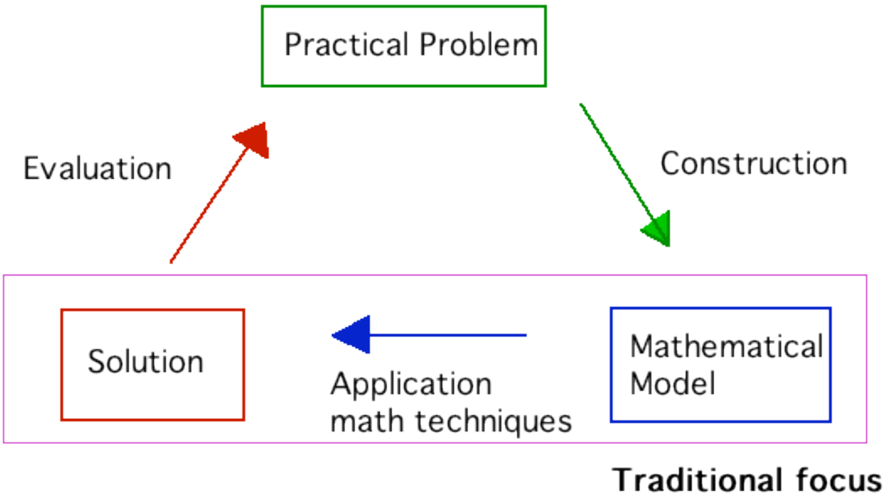


Figure 1

Since the real world is complex and varied, while mathematics deals with ideal, abstract objects, simplifying assumptions always have to come into play when mathematics is applied. In the real world there are no perfectly straight lines, flat planes or spheres nor can measurements be made with absolute precision. Even an elementary arithmetic sum like "2 apples plus 1 apples = 3 apples" is tacitly based on the implicit assumption that these apples are ALL EXACTLY the SAME (in size, shape, and that none of them are rotten!) As shown in Figure 2, the three vertical groups of the "*correct answer*", 3 apples, are hardly equivalent at all!

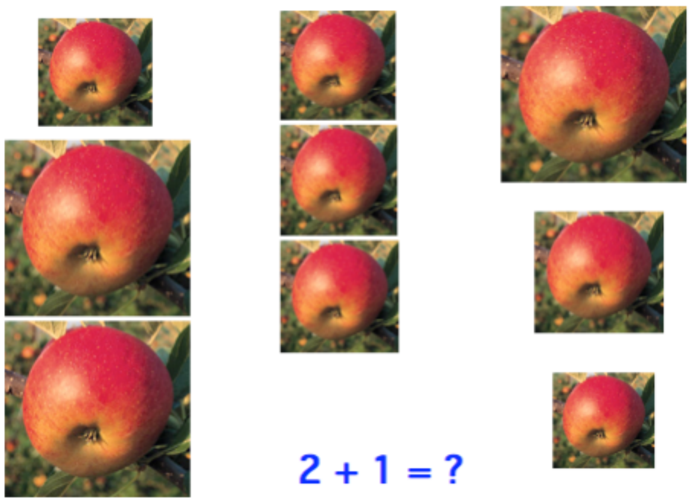


Figure 2

To what extent are we making learners aware and critical of some of the implicit assumptions in many so-called practical applications of mathematics? Not doing so certainly cannot help learners' better distinguish between mathematics and the real world, nor help their understanding of the inter-relationship between the two.

Another example is the following typical problem from trigonometry from the FET phase:

> *A person on a cliff 100m high sees a ship out at sea. If the angle of declination is 30 degrees, how far out at sea is the ship?*

Normally it is expected of learners here to make a little sketch as shown in Figure 3, and then use the tan function to find the distance from the base of the cliff to the ship. A nice simple application of trigonometry, beautifully illustrating the utility of mathematics, isn't it?

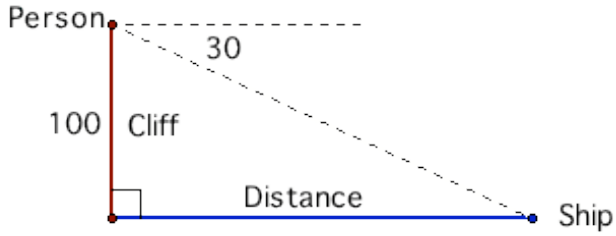


Figure 3

But what are some of the assumptions that have been made in this problem and its solution? Clearly, there are several over-simplifications of reality, for example:

1. It is assumed that the ocean is flat, i.e. that a straight line can reasonably approximate the distance from the base of the cliff to the ship. In other words, the curvature of the earth is ignored, which on a small scale is not problematic, but on a larger scale may lead to significant deviations.
2. It is assumed that the cliff is perfectly vertical with respect to the line chosen to represent the distance from the base of the cliff to the ship.
3. More over, it is assumed that the person's height (of say 1.7 m) is negligible compared to the height of 100 m of the cliff.
4. It is also assumed that the ship is far enough away that a single point can reasonably represent its position, which is certainly not the case for a large ship closer to the shore.

Now these are not unreasonable assumptions to make to simplify the problem, and thus to obtain a reasonable ESTIMATE of the distance. However, mathematically educated learners ought to know that the calculated answers we obtain for applied problems such as these are NEVER absolutely precise (leaving out even the issue of how accurately the angle of declination was measured).

Furthermore, by letting learners find a more accurate solution by using a more accurate representation of reality, may help develop understanding that usually the gain in accuracy is offset against increased mathematical complexity. This happens quite often in mathematical modeling in various fields such as physics, biology, engineering, statistics, etc. so that simpler models are quite often chosen because they are easier to work with mathematically, though one should always be aware of the assumptions and therefore the limitations that they may potentially have.

Stages 2 & 3

After obtaining some kind of mathematical model to represent reality, we normally proceed with the solution process, which can involve simple arithmetic operations or more advanced ones like factorization, solution of equations, differentiation, transformations, etc. Lastly, in the interpretation and evaluation of the solution we need to check whether it is realistic by critically comparing it with the real world situation.

Let us now briefly consider and discuss Stages 2 and 3 in relation to the following four problems that I've also regularly over the years given to prospective primary mathematics teachers to try out with their learners during teaching practice, namely:

1. *A group of 100 people have to be transported in minibuses. No more than 12 passengers may be transported in one minibus. How many minibuses are needed?*
2. *Packages of 12 cookies each have to be made up for a bazaar. How many packages of 12 can be made up if 100 cookies are available?*
3. *In a famine stricken area, 100 pockets of rice have to be shared equally between 12 families. How many pockets of rice should each family get?*
4. *100 people have to be seated at 12 identical tables. How many people are to be seated at each table?*

As before, it is quite typical of older children in Grades 5, 6 and 7 who have been taught and drilled extensively in long division, to quickly recognize that all four problems can mathematically be represented by the number sentence 100 ÷ 12. Shockingly, however, the majority of them would then usually just offer the answer "8 remainder 4" for all four these problems, showing no appreciation whatsoever that such an "answer" is entirely SENSELESS in each of the four problems. Such learners have effectively become mathematically DISEMPOWERED, contrary to any definition of education!

In contrast, younger children often model and solve these four problems quite sensibly, often using a variety of different ways. For example, Problem 1 could be solved by a young child by repeated addition, by adding up the number of people each taxi can take one after the other, e.g. 12 + 12 => 24 +12 => 36 + 12 => 48, etc. until 96 is reached. The child then typically adds up the number of 12's to find the number of taxis, which is 8, and then adds one more taxi for the remaining 4 people. So the required number of taxis is 9.

(Something one could raise further in a class discussion, is that if additional cost was not an issue, one might as well use 10 taxis to transport 10 people per taxi and which would be more comfortable than having 12 people per taxi. Of course, one could also argue that the remaining 4 people should just be squeezed in, for example by putting one extra person in 4 of the 8 taxis. But then one is raising important issues of road safety and breaking the law by overloading! Even if the class decides that 9 mini-buses will suffice, a decision will still have to be made whether one is going to transport 12 people in each of the 8 taxis, and 4 people in 1 taxi OR whether one is going to transport 11 people in each of 8 taxis plus 12 people in one 1 taxi. From the viewpoint of a little more 'comfort" for more people, the latter solution may be favored).

This example clearly shows how a solution in the final stage of interpretation and evaluation can be influenced by several important real world considerations. The other three problems above can be solved similarly with possible respective answers as follows:

2. 8 packages of 12 (here the remainder of 4 cookies could be eaten or put in a smaller packet and maybe sold at 1/3 of the price?)
3. $8\frac{1}{3}$ pockets of rice per family (since it is a famine stricken area one would want to divide up the 4 remaining pockets among the 12 families so that each gets 1/3)
4. by seating one extra person at 4 of the tables, we obtain 4 tables with 9 people and 8 tables with 8 people (note how a typical answer of 8 rem 4 is particularly nonsensical for this problem).

Mere computational skill at long division clearly does not guarantee learners obtaining sensible and meaningful solutions to these four problems. Even using a calculator to carry out the "division" and obtain an answer of 8.33333333 is basically useless without proper interpretation within the given practical contexts. But are our children allowed sufficient opportunities such as these to engage in stage 3 of the modeling process?

Let us also briefly consider again Sally's doll problem from a modeling perspective:

Sally has some dolls and is given 5 more dolls by her grandmother so that she now has 12 dolls in total. How many dolls did she have in the beginning?

As mentioned earlier, many younger children using less sophisticated calculation strategies (e.g. drawing pictures, using physical objects, counting all, or counting on, etc.) are often less likely than older children to simply compute 12 + 5 as the answer. Closer analysis reveals that one explanation seems to be found in the prescriptive rule-oriented teaching of some teachers.

Where-as the actual number sentence or mathematical model that best represents this problem is ? + 5 = 12, some teachers apparently INSIST that learners MUST write down a number sentence for this problem in the form 12 – 5 = ? These teachers are clearly confusing one possible SOLUTION STRATEGY, namely 12 – 5 = ?, with that of first accurately MODELING the situation. The number sentence 12 –

5 = ? is no reasonable representation of the problem situation at all – the real situation does NOT involve any "subtraction", i.e. any "taking away" or "decrease".

In fact, the situation involves an "increase" in the number of dolls, which children readily recognize, but because they are forced to write number sentences with the unknown only on the right side, they now have no alternative but to write 5 + 12 = ?, and proceed to carry out the calculation. Here is a clear example of INTERFERENCE with children's correct intuitive perception of a situation by their following a teacher's prescribed rule as a result of their natural desire to please him/her.

Furthermore, note that a reasonable model for this problem need not be a number sentence, but can also be drawings, counters, fingers, etc. In solving such models, children may also proceed in a number of different ways, for example:

- counting on (**6**, **7**, **8**, **9**, **10**, **11**, **12** => 7)
- piecemeal addition ($\underline{5}$ + 5 => 10 + $\underline{2}$ => 12; thus $\underline{5}$ + $\underline{2}$ = 7)
- trial & error addition ($\underline{10}$ + 5 = 15, too big; $\underline{6}$ + 5 = 11, too small; $\underline{7}$ + 5 = 12)
- counting backwards (**11**, **10**, **9**, **8**, **7**, **6**, **5** => 7)
- subtraction (12 – $\underline{2}$ => 10 – $\underline{3}$ => 7)

Unfortunately teachers often think they are HELPING learners by providing them with rules like these, but the sad fact is that learners are being helped into their "mathematical GRAVES"! Prescriptive teaching generally tends to destroy children's natural creativity and ability to solve real world problems (as well as purely mathematical ones) meaningfully. What happens is that children become so pre-occupied with trying to comply to the required rules of the teacher, that they no longer think about the actual problem at hand nor try to solve it in any way they can. Another consequence of prescription teaching is that children become more and more teacher-dependent rather than independent, expecting all the time to be first shown an example of how a problem of a certain type should be solved, before dutifully practicing several similar examples.

The role of technology

As discussed and illustrated with several examples in De Villiers (1994), computing technology can assist immensely with particularly the SOLUTION stage of the modeling process. Apart from calculators having the ability to now perform complicated arithmetic calculations with speed, ease and accuracy, symbolic algebra software like *Mathematic*a and *Maple* that can easily solve equations, factorize expressions, differentiate, etc. are now widely available too. In addition, dynamic geometry software like *Sketchpad* enables one to easily simulate and solve problems by animation and/or using scale diagrams.

Computing technology therefore strongly challenges the traditional approach, which emphasizes computational and manipulative skills BEFORE applications (and usually at the cost of developing skills in model construction and interpretation). For example, Grade 9 or 10 learners can now solve a traditional calculus problem like the following simply by using a calculator and completing a table of values as in Makae et al (2001) or better still use the graphing capability of *Sketchpad* as shown in Figure 4:

> *A water reservoir in the shape of a rectangular prism with a square base has to be built for three villages and needs to contain 80 000 liters. What should its base-length and height be to use the least amount of material?*

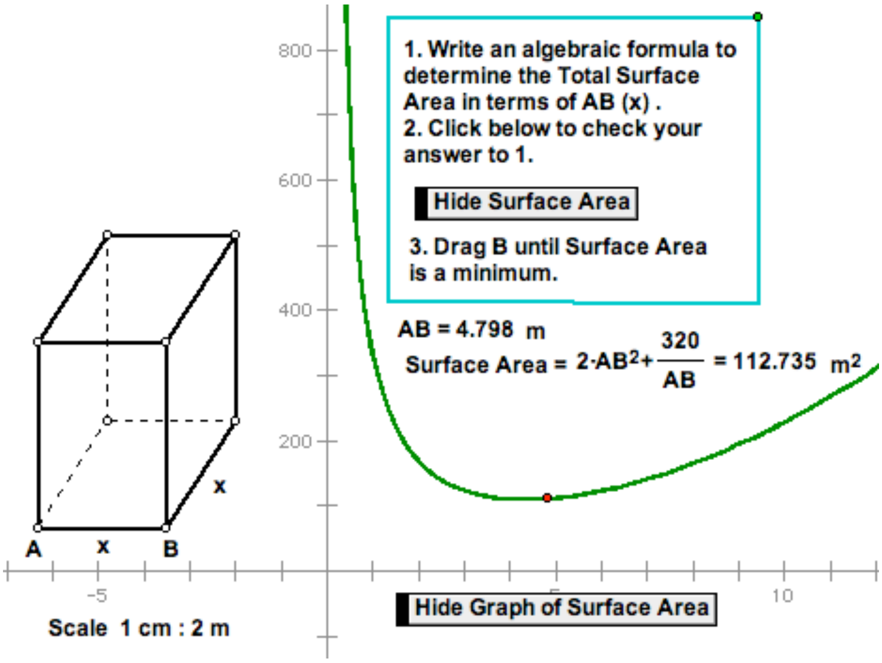


Figure 4

However, note that in order to use computing technology effectively in this way, it is essential to first obtain an appropriate algebraic model, i.e. the surface area function. Older learners in Grade 12 can even differentiate this surface area function within *Sketchpad*, draw a graph of the differential function, and then find the minimum where it cuts the *x*-axis. A heavy emphasis on computation and manipulation BY HAND in mathematics education can therefore no longer be justified with the increased availability of such software, not only on computers, but also on graphic calculators.

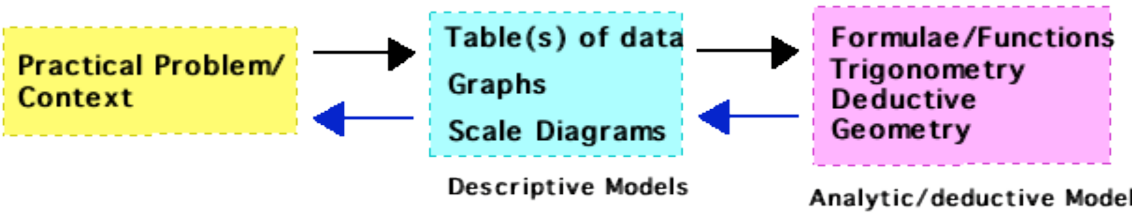


Figure 5

With computing technology, modeling as a teaching approach becomes feasible as illustrated in Figure 5. For example, the following alternative to the traditional "theory-first – applications-later" approach to quadratic functions can now be used:

Stage 1: Where do quadratic functions come from?

* Falling Objects, Projectile Motion & other Scientific contexts
* Non-linear revenue & other Economic contexts
* Curve-fitting (e.g. least squares)

Stage 2: Finding max/min values & solving quadratic equations

* Guess-and-test
    * By hand: numerically & graphically
    * By calculator: numerically & graphically
    * By numerical methods such as iteration: by hand & calculator
    * By computer: numerically & graphically
* Formal Solutions by Computer
    * Symbolic Processors like *Maple*, *Derive*, *Mathematica*, etc.

Stage 3: Formal Theory of Quadratic Functions

* Factorization
* Completion of square
* General solution formula
* Discriminant
* Axis of symmetry, maximum/minimum
* Sum & product of roots

Simulation and animation

Apart from greatly assisting in the Solution Stage of modeling, computing technology also provides useful modeling tools for simulation and animation. For example, consider the following problem: What is the path of the head of a man's shadow if he is walking past a lamppost AB, in a straight line? As shown in Figure 6 this can be easily modeled, and a solution dynamically found by moving *CD* along the straight path, and observing the path that *E* traces out. (Of course, this provides no formal proof that the top of the shadow also moves in a straight line parallel to the straight path the man is walking on, but knowing what one needs to prove is often a necessary first step. In this case, the result can be proved using the properties of similar triangles, and is left to the reader.)

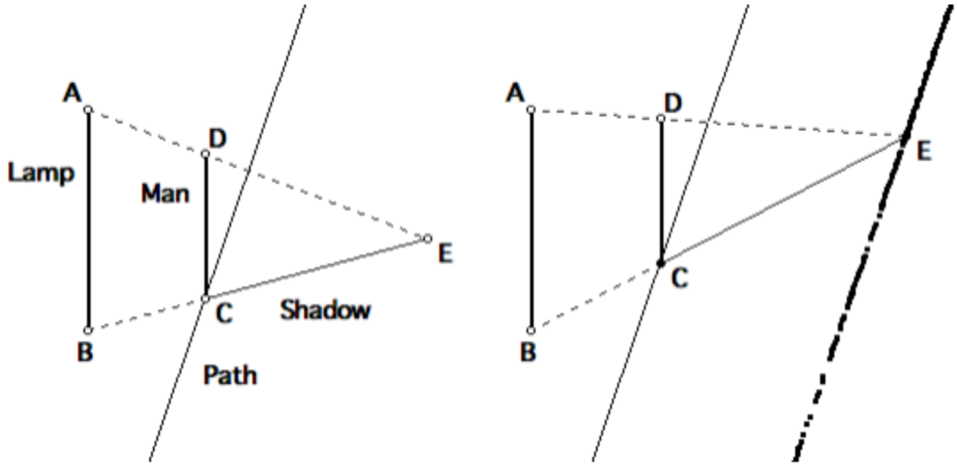


Figure 6

Similarly, as discussed in De Villiers (1999), *Sketchpad* can very effectively be used to dynamically investigate the best place for kicking to the posts in rugby after the scoring of a try as shown in Figure 7.

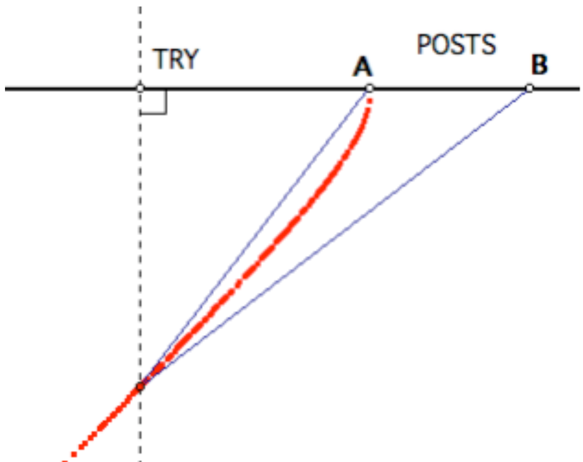


Figure 7

Figure 8 shows how the following interesting problem can also be modeled and solved: How large must a mirror be so that you can see your whole body in it?

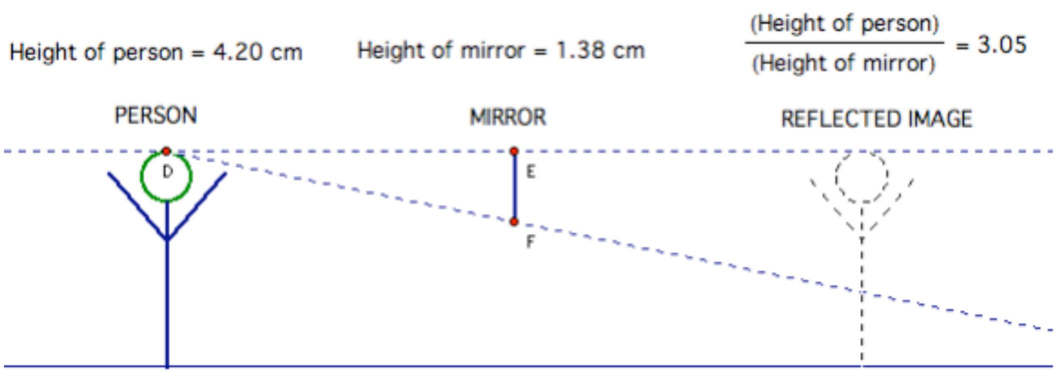


Figure 8

Through visual animation, *Sketchpad* can also bring to life the topic of solving simultaneous linear equations. For example, learners could use a sketch as shown in Figure 9 to investigate which of Henri or Emile would win a race of 150 m, if they have respective running speeds of 2.5 m/s and 1 m/s, and Henri has a head start of 45 m. To enhance the connection with the underlying mathematics linear graphs of their respective distances against time can be drawn in the same sketch. Changing the speeds or head start given to Henri further allows learners to dynamically investigate a whole range of different related problems.

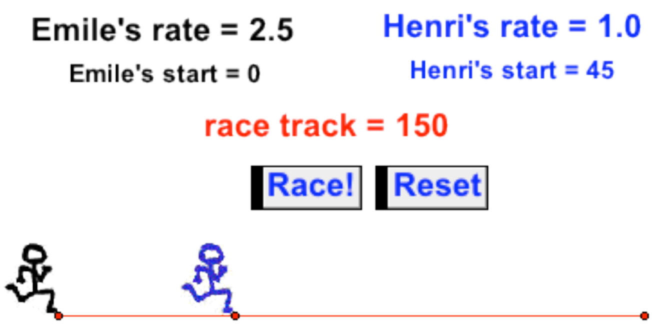


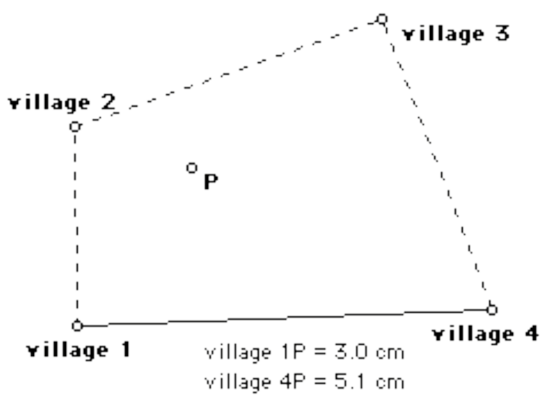
Figure 9

An example of modeling a geometric concept

Traditionally the concept of perpendicular bisector is just introduced by definition as a "line that perpendicularly bisects a line segment". This is usually followed by a description of how one can construct a perpendicular bisector of a line segment by compass, and perhaps a proof using congruency that the construction actually works. Next follows the theorem that the perpendicular bisectors of a triangle are always concurrent (and that the point of concurrency is the circumcentre of the circumcircle of the triangle). Apart from the fact that FET learners are no longer required to prove this result, no effort is usually made to connect this result to any real world context.

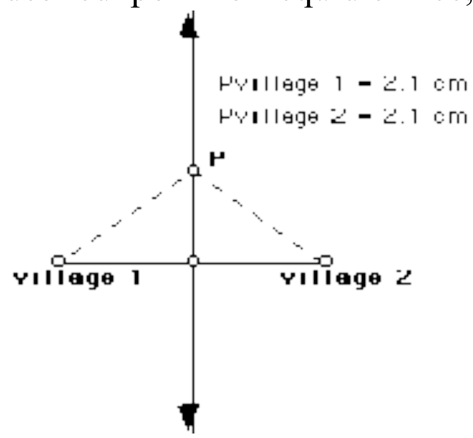
In De Villiers (2003) the concept of perpendicular bisector is NOT presented directly at first, but learners and prospective teachers are first given the following realistic problem to investigate with a ready-made sketch in *Sketchpad*:

In a developing country like South Africa, there are many remote rural areas where people do not have access to safe, clean water, and are dependent on nearby rivers or streams for their water supply. Apart from being unreliable due to frequent droughts, these rivers and streams are often muddy and unfit for human consumption. Suppose the government wants to build a water reservoir and purification plant for four villages in such a remote, rural area. Where should they place the water reservoir so that it is the same distance from all four villages?



After first using trial and error and dragging *P* until it is EQUIDISTANT from all four villages, learners and prospective teachers are asked to try and find a different, easier method of locating the desired position. In order to do this, they are encouraged to try and find all the equidistant points from two of the villages, and using the drag and TRACE function of *Sketchpad*, they find that all these equidistant points form a line bisecting the line segment connecting the two villages (and perpendicular to it). So here the concept of perpendicular bisector of a line segment is DEFINED as the "path (or locus) of all the points equidistant from the two endpoints of the segment".

After constructing the four perpendicular bisectors of the sides of the quadrilateral to note that they all intersect in the desired point of equidistance, learners and prospective teachers are then asked whether one can always find a point equidistant from all four vertices, no matter the shape or size of the quadrilateral. Since most learners and prospective teachers usually expect it to be possible, they then find it SURPRISING when dragging the vertices of the quadrilateral that the perpendicular bisectors are no longer concurrent; i.e. in other words, that not all quadrilaterals have equi-distant points from their vertices!



Having learners and prospective teachers construct a circle with its centre at the circumcentre (the point of concurrency) of the original quadrilateral, and passing through all four villages, the concept of "cyclic quadrilaterals" is now also introduced in a practical, meaningful context. (In contrast to the decontextualised way it is normally introduced in Grade 12).

More-over, when we now move on to next investigate where a water reservoir should be placed so that it is equal distances from three villages, it now comes as a big SURPRISE when DRAGGING, that for a triangle, no matter the shape or size, the perpendicular bisectors always remain concurrent! As explained and discussed in Mudaly & De Villiers (2004), this surprise can now be effectively utilized to introduce proof to learners as a means of "explaining" why this result is always true.

Part of both activities also requires learners and prospective teachers to identify what assumptions have been made to simplify the problems that may not necessarily be true in the real world, and how real world factors such as terrain or size of the villages, may affect the desired solution(s). Consideration is also given to examining the interesting (and challenging) situation of where to put the water reservoir for a quadrilateral that doesn't have concurrent perpendicular bisectors (i.e. a non-cyclic quadrilateral).

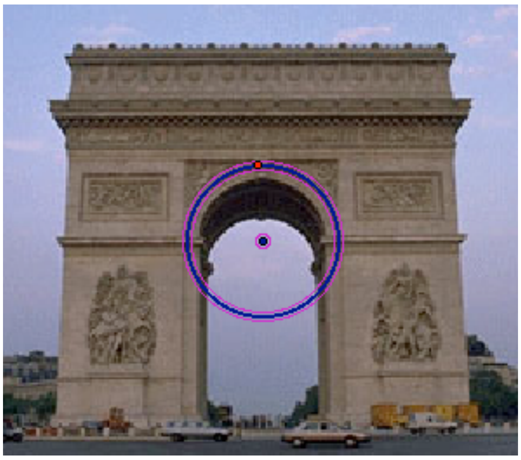


Figure 10

Dynamic modeling of real world objects

Lastly, since one can paste pictures directly into *Sketchpad*, one can easily use it to investigate the geometric properties of real objects from their photographs. For example, Figure 10 shows how the half-circular arcs of the Arc de Triomphe in Paris can be modeled and illustrated easily. Similarly, it is shown in Figure 11 how the arch of the Tollgate Bridge in Durban can be modeled reasonably well using either a parabolic or exponential function.

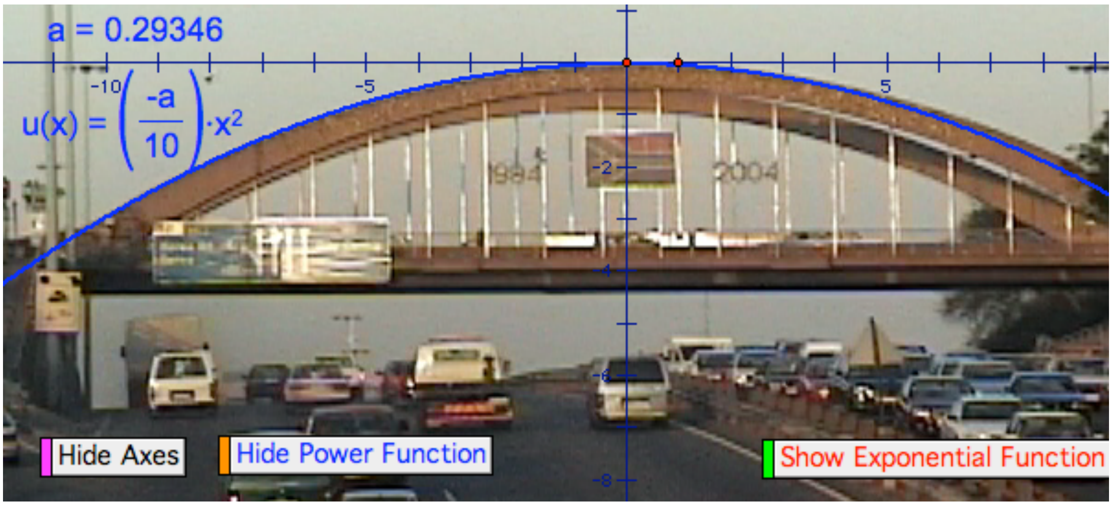


Figure 11

References

De Lange, J. (1997). Egte probleme met realistiese wiskunde, (shortened translation of ICME 8, Plenary, Real problems with Realistic Mathematics), **Pythagoras**, Aug, No. 43, pp. 5-12.

Dept. of Education. (2002). Draft National Curriculum Statement: Mathematics. Pretoria.

De Villiers, M. (1987). Teaching Modeling and Axiomatization with Boolean Algebra, **The Mathematics Teacher**, Vol.80, No.7, pp.528-532.

De Villiers, M. (1992). Relevant, contextualized teaching versus irrelevant, decontextualised teaching. **Pythagoras**, Aug, No. 29, p. 3.

De Villiers, M. (1994). The role of technology in mathematical modeling. **Pythagoras**, Dec, No. 35, pp. 34-42. (At http://mysite.mweb.co.za/residents/profmd/techmod.pdf)

De Villiers, M. (1999). Place-kicking locus in rugby. **Pythagoras**, Aug, No. 49, pp. 64-67. (Available at http://mysite.mweb.co.za/residents/profmd/rugby.pdf)

De Villiers, M. (2003). **Rethinking Proof with Sketchpad 4**. Key Curriculum Press, USA.

Kowlesar, S. (1992). An investigation into Grade 11 pupils' understanding of linear functions in real world contexts. Unpublished B.Ed (Hons) thesis, University of Durban-Westville.

Makae, E.J, Nhlapo, M. et al. (2001). **Let's Talk Mathematics!** (Grade 9 Mathematics Textbook), Juta-Gariep Publishers.

Murray, H., Olivier, A. & Human, P. (1998). Learning through problem solving. In A. Olivier & K. Newstead (Eds.), **Proceedings of the Twenty-second International Conference for the Psychology of Mathematics Education: Vol. 1.** (pp. 169-185). Stellenbosch, South Africa.

Van Etten, B. & Adendorff, S. (2006). A brief Analysis of the Grade 9 CTA in 2005. **Learning and Teaching Mathematics**, Dec, No. 4, pp. 39-44.

Mudaly, V. & De Villiers, M. (2004). Paper presented at 10th AMESA Congress, 30 June -4 July 2004, University of the North-West, Potchefstroom. (Available at: http://mysite.mweb.co.za/residents/profmd/vimmodel.pdf)