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Flying Two Kites: Part 1

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When you classify triangles as scalene (no sides equal); isosceles (two sides equal) and equilateral (three sides equal) you are describing an inheritance structure where a triangle further down the list inherits all the properties of the one above it as well as having something unique of its own. Using the same classification structure referring to angles, we get a scalene triangle (no equal angles), an isosceles triangle (two equal angles) and an equilateral triangle (three equal angles). There is a duality between *side length* and *angle*. The preferred method seems to be the classification according to relative length rather than relative angle and that's probably because Euclid went about it that way. Angle size classification (obtuse, right, acute) does not reflect the dual "level" of symmetry of triangles.

A best set of definitions for triangles or quadrilaterals would imply other properties in a symmetric manner and reveal the symmetries of the polygons. For triangles two equal sides implies two equal angles, and conversely. The same applies for three equal sides. We get two properties for one because of this duality. A triangle where a median is coincident with an angle bisector is also a possible definition for an isosceles triangle. An equilateral triangle can be similarly defined as a triangle where two medians are coincident with two angle bisectors. The third pair is superfluous as a definition should only have necessary and sufficient, not a litany of properties. Where two pairs go the third follows.

We can define a convex quadrilateral as having diagonals that intersect within the figure and concave as having diagonals, which intersect outside the figure. We could also classify according to angle size. Concave quadrilaterals have one (interior) angle greater than 180 degrees but convex quadrilaterals have all interior angles less than 180 degrees. Although both are arbitrary, the first definition seems neater. One should not confuse a definition with a property. Many student exercises in quadrilateral classification consist of writing a column of properties and a row of quadrilateral shapes, both in any order, and placing ticks or crosses in the corresponding grid. Is there a better way?

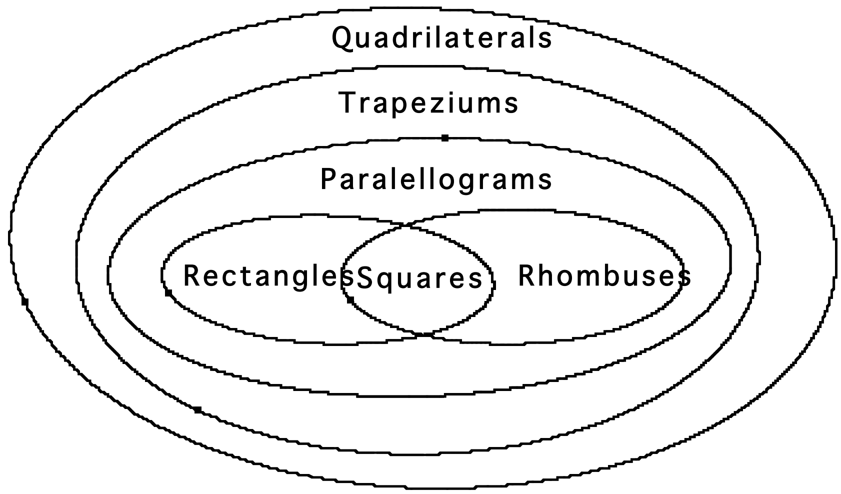

Figure 1

The Australian N.S.W. Grades 7-10 Mathematics Syllabus on page 143 suggests the Venn diagram classification scheme for quadrilaterals shown in Figure 1. The following accompanies the classification scheme: "*When classifying quadrilaterals students need to begin to develop an understanding of the inclusivity of the classification system. That is trapeziums are inclusive of the parallelograms, which are inclusive of the rectangles and rhombuses, both which are inclusive of the squares*." Is their classification appropriate? The context would indicate we are dealing with convex quadrilaterals. What happened to the kites? There are, after all, two classes or types of kite and two classes of trapezium.

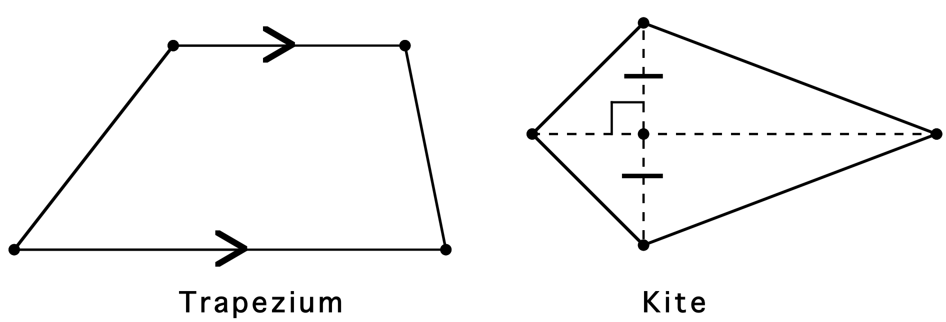

Figure 2

Figures 2a and 2b are what is generally called a trapezium and a kite in N.S.W. Schools. The diagrams themselves illustrate the properties some would usually use as definitions.

The trouble is the trapezium has one simple defining property and the "*isosceles kite*" (my preferred term) has two, one diagonal is bisected and the diagonals are also perpendicular. Others might define a kite as a convex quadrilateral with two pairs of adjacent sides equal. That's still two properties. It's still two properties if we plumb for one pair of opposite angles equal as we need another constraint to make it an isosceles kite. This is a reflection of the fact that the two quadrilaterals illustrated below are on different "levels" of symmetry. The sub class of trapezium where the two non-parallel sides are equal is referred to as an "*isosceles trapezium*". It is on the same level of symmetry as an "*isosceles*" kite.

Thus, I personally prefer to consider a "*kite*" as a convex quadrilateral with the property that one diagonal is bisected but not at right angles as a working definition. One could call it an "*oblique kite*" or a "*scalene kite*" if that allows a consistent use of language. De Villiers (1996: pp. 154-155; 206-207) calls this a "*bisecting quad*". I refer to the sub class illustrated below as an isosceles kite just as an isosceles trapezium is named. The sub class of trapezium where the two non-parallel sides are equal is referred to as an isosceles trapezium and it is on the same "level" of symmetry as an "*isosceles*" kite.

The scheme in Figure 3 ignores the general kite and general trapezium. The diagram represents the limit of our angle-side duality. The parallelogram has point symmetry but no axis of symmetry. The isosceles trapezium and isosceles kite both have an axis of symmetry but no point symmetry. The rectangle and rhombus both have two axes of symmetry and point symmetry. The square has four axes of symmetry and point symmetry. This may be a yr 7 or 8 school schema. It doesn't tell the whole truth but it does not lie.

If we take on the general (or scalene) trapezium we need to take on the general (or scalene) kite in order to keep the structure logical and symmetric. We need to be very attentive to keep our definitions in angle-line form and consider defining a parallelogram as a structure inheriting all the properties of (scalene) kite and (scalene) trapezium. If we pick the wrong pair we get the wrong shape. This emphasizes the fact that the parallelogram is on the same symmetry level as the isosceles kite and the isosceles trapezium. Even the definitions of (scalene) kite and (scalene) rhombus seem forced.

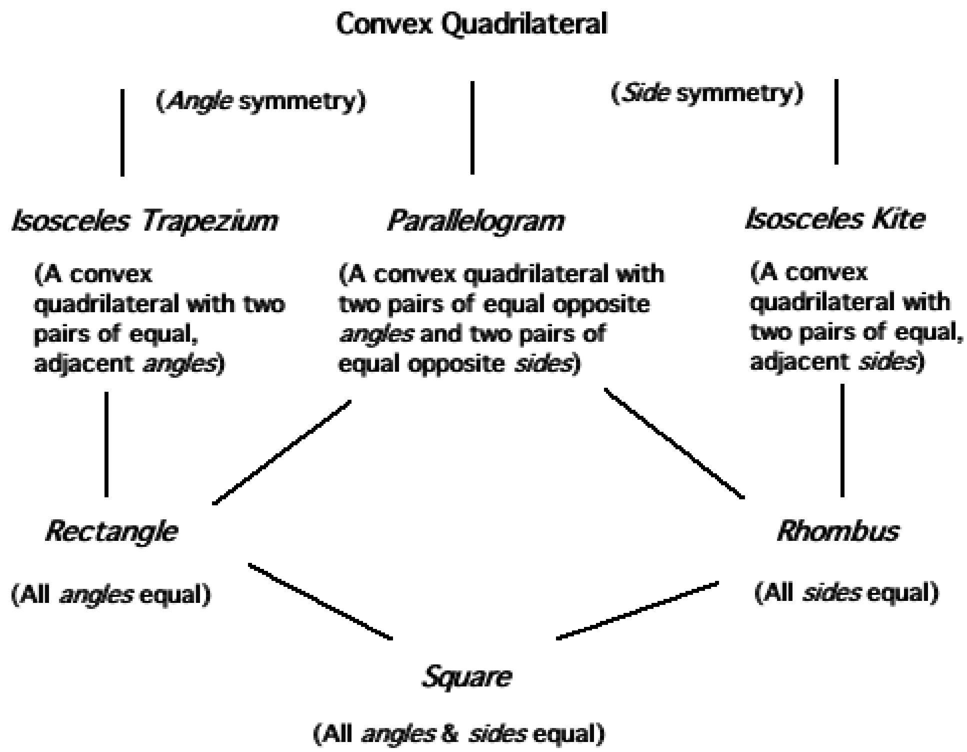

Figure 3

By looking at diagonal properties we see the same symmetry structure as shown in Figure 4. The structure, not necessarily the definition set, is important. The definitions are themselves duals and the symmetry is compelling. It may be possible to use inheritance in an object-oriented sense. The use of duality and an inheritance taxonomy allows for rapid assimilation of the properties as well as a scaffolding structure for important mathematical ideas. The number of properties increases as we move down the list but inheritance tells us that if the property holds for a parent class it holds further down and so we need not list them all. There are properties, which do not have a dual. I have thrown in the cyclic and circum quadrilaterals in the next issue and there I propose an inheritance structure based on medians and diagonals only.

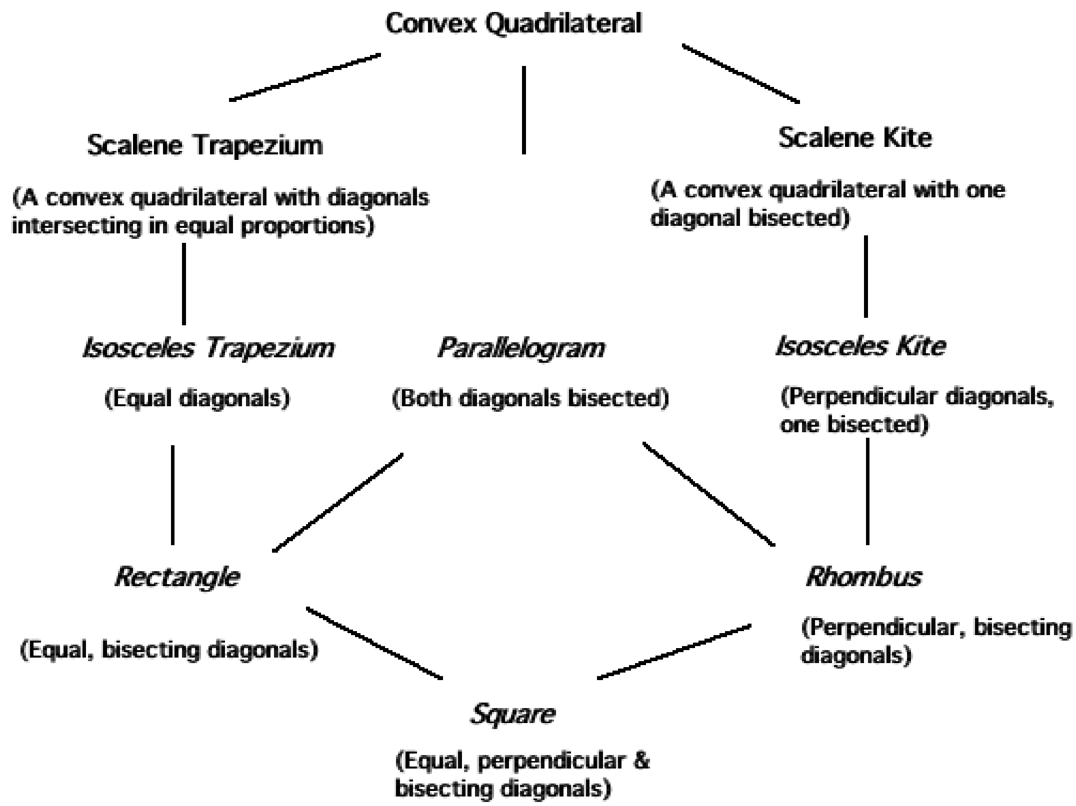

Figure 4

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"For some years I had noted that the effect of the early introduction of arithmetic had been to dull and almost chloroform the child's reasoning faculties. There was a certain problem which I tried out, not once but a hundred times, in grades six, seven, and eight. Here is the problem: "If I can walk a hundred yards in a minute [and I can], how many miles can I walk in an hour, keeping up the same rate of speed?" In nineteen cases out of twenty the answer given me would be six thousand, and if I beamed approval and smiled, the class settled back, well satisfied. But if I should happen to say, "I see. That means that I could walk from here to San Francisco and back in an hour" there would invariably be a laugh and the children would look foolish."

- L. P. Benezet,*The Journal of the National Education Association*, Volume 24, Number 8, November, 1935, pp. 241-244

Flying Two Kites: Part 2

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"*There is no permanent place in the world for ugly Mathematics*." G.H. Hardy

A classification scheme for convex quadrilaterals based on diagonal and median properties is now given in Figure 5. Note that a *median* of a quadrilateral is defined as the segment joining the midpoints of opposite sides. The properties, which can be used to defined the corresponding quadrilaterals, reveal the duality as well as the inheritance structure within this classification scheme.

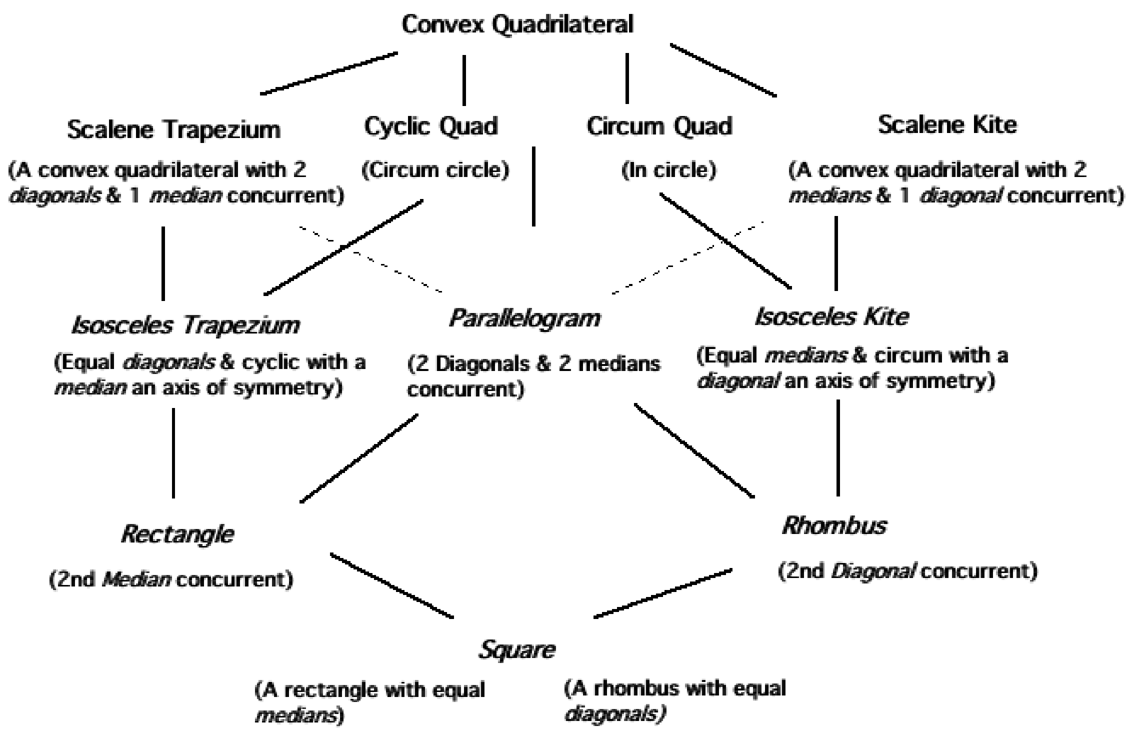

Figure 5

Some properties of the quadrilaterals in Figure 5 are tabulated below.

Scalene Trapezium (Trapezium)	**Scalene Kite (Kite)**
Diagonals intersect in equal proportions.	One diagonal bisects the other.
Diagonals and one medium are concurrent.	Medians and one diagonal are concurrent.
Any line segment parallel to one median is bisected by the other median. (True for all	Any line segment parallel to the bisected diagonal is bisected by the other diagonal.

quadrilaterals.)	Cavaleri's principle implies the next property.
The area on one side of the median is the same as the area on the other. (True for all medians on any quadrilateral.)	The area on one side of the diagonal, which is not bisected, is the same as the area on the other.

Cyclic Quadrilateral	**Circum Quadrilateral**
Circumscribed circle.	Inscribed circle.
Perpendicular bisectors of the sides are concurrent at the circumcentre.	Angle bisectors of the angles are concurrent at the incentre.
The sums of the two pairs of opposite angles are equal.	The sums of the two pairs of opposite sides are equal.

Isosceles Trapezium	**Isosceles Kite**
One pair of opposite sides equal.	One pair of opposite angles equal.
Two pairs of equal adjacent angles.	Two pairs of equal adjacent sides.
Has a circumscribed circle.	Has an inscribed circle.
An axis of symmetry through one pair of opposite sides. (One Medium)	An axis of symmetry through one pair of opposite angles. (One Diagonal)
Medians are perpendicular.	Diagonals are perpendicular.
One median is a perpendicular bisector.	One diagonal is an angle bisector.
The diagonals are congruent.	The medians are congruent.

Parallelogram	**Parallelogram**
The opposite angles are congruent.	The opposite sides are congruent.
Both diagonals are axes of affine symmetry.	Both medians are axes of affine symmetry.
The medians bisect each other.	The diagonals bisect each other.
Adjacent side pairs have the same total length.	Adjacent angles are supplementary. (Adjacent angle pairs have the same total angle.)

Rectangle	**Rhombus**
All angles are equal.	All sides are equal.
An axis of symmetry through each pair of opposite sides. (Median symmetry)	An axis of symmetry through each pair of opposite angles.(Diagonals symmetry)
The medians are perpendicular.	The diagonals are perpendicular.

Square	**Square**
All angles are equal.	All sides are equal.
The diagonals bisect each other.	The medians bisect each other.
An axis of symmetry through each pair of opposite sides. (Median symmetry)	An axis of symmetry through each pair of opposite angles.(Diagonals symmetry)

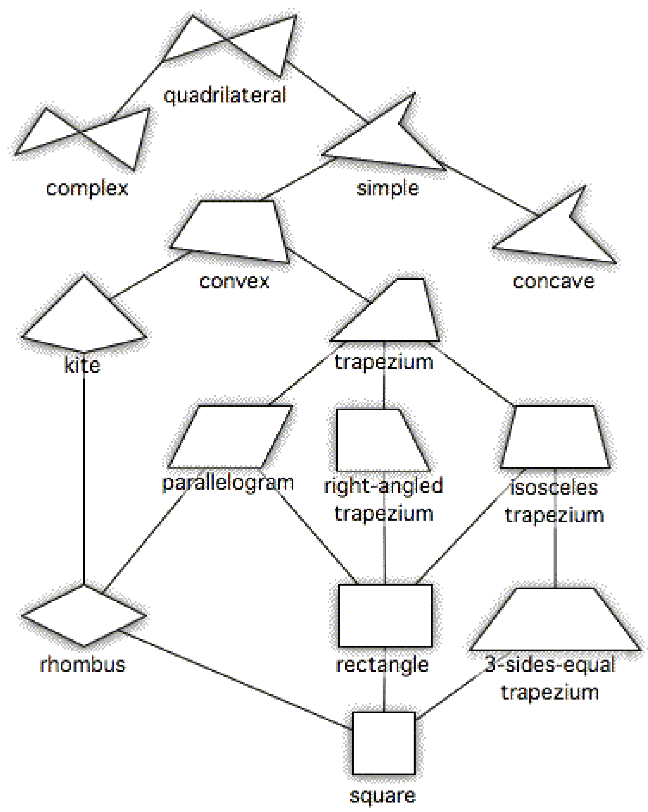

Figure 6

One should note there are many possible classification schemas, some even much more extended that includes "*crossed*" quadrilaterals.[1] However, not every possible quadrilateral need be included in an inheritance structure. Figure 6 shows an example from a website, which does NOT make any use of symmetry and proposes additional shapes. The 3-sides equal trapezium and right-angled trapezium are unnecessary additions if definitions are not side-angle based.

Why are we "drawn" to definitions based only one side length? Is it because of how we usually construct or draw quadrilaterals or triangles? Did the Greeks have a reason for banning the (marked) ruler?

Note

An extended classification of simple closed and crossed quadrilaterals based on the side-angle duality can be downloaded from:

http://mysite.mweb.co.za/residents/profmd/quadclassify.pdf

References

(a) Figure 1 The current N.S.W. year 7 to 10 mathematics syllabus.

(b) Figure 6 http://en.wikipedia.org/wiki/Quadrilateral (A counter example)

(c) An Interesting Duality in Geometry, (found while writing this article)
http://mysite.mweb.co.za/residents/profmd/amesa96a.pdf

(d) De Villiers, M. (1996). *Some Adventures in Euclidean Geometry*, Univ. of Durban-Westville (now Univ. of KwaZulu-Natal)

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"Life without geometry is pointless." - Anonymous

"*... the Great Bear is looking so geometrical. One would think that something or other could be proved*." - Christopher Fry