A Merry-Go-Round the Triangle

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> "*The reasonable man adapts himself to the world; the unreasonable one persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable man*." – George Bernard Shaw

> "*I hear you say 'Why?' Always 'Why?' You see things; and you say 'Why?' But I dream things that never were; and I say 'Why not?*'" – George Bernard Shaw

The above quotations provide wry yet intriguing descriptions of 'out of the box' thinking, and how sometimes being 'unreasonable' can actually be productive. Asking the simple question 'why not?' in mathematics can often lead to the discovery and posing of new conjectures and problems. For example, suppose a particular result holds for an equilateral triangle. A first step may be to ask: 'why not' explore the same idea for a square, a regular pentagon, etc.? Similarly, 'why not' explore whether it also holds for an isosceles triangle or perhaps *any* triangle? 'Why not' explore the idea in 3D space? 'Why not' vary the idea itself? In a similar vein, Brown and Walter (1990) highlight the importance of 'what if not?' questions. What if it were not a triangle in the plane, but instead on a curved surface such as a sphere?

This kind of thinking lies at the heart of a rather neglected area of mathematics education, namely the development of *problem posing*. In order to pose new problems it helps to be able to think divergently and ask questions that interrogate and vary the givens and conclusions of a result. The purpose of this article is to give an example of how this thinking led to the formulation of a new conjecture (for the authors at least) along with several associated results as well as their proof.

THE STARTING POINT

A well-known theorem in geometry states that if one starts at the midpoint of one of the sides of a triangle, and draws a line parallel to either of the other two sides, then that line will bisect the third side (see Figure 1a).

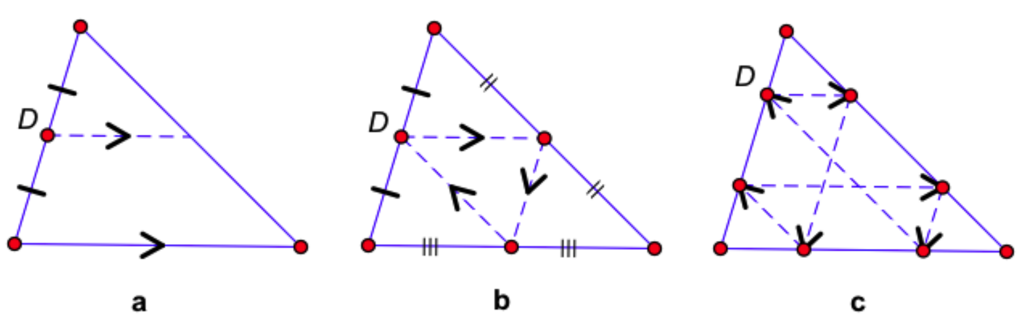

FIGURE 1

Continuing drawing parallel lines in this way produces the so-called median triangle by joining the midpoints of the sides (see Figure 1b). But *what if* we start at point D when it is *not* at the midpoint, and continue drawing parallel lines in the same way as before? Will we ever come back to the starting point or will the process carry on forever? The experience of one of us (De Villiers) with high school learners as well as university students has been that they usually find it surprising that one comes back to the starting point after drawing only six parallel lines, no matter where D is chosen (see Figure 1c). It is really nice to visually experience this by dragging D in a dynamic geometry sketch. This result appears in De Villiers (2009, 2012) and can easily be proved by the theorem which states that if a line from a point on a side of a triangle is drawn parallel to another side, then that line will divide the remaining side in the same ratio that the original starting point divides the original side.

CONJECTURING

Recently one of us (De Villiers) was looking at the result illustrated in Figure 1c and realized that another way to think about it was that one could view the lines as being drawn parallel to the sides of the median triangle. Immediately it seemed natural to ask the question *what would happen if* the median triangle were replaced by the feet of an arbitrary trio of concurrent cevians, i.e. by a Cevian triangle[2]. A quick construction and test on *Sketchpad,* as shown in Figure 2, revealed the following interesting result: "Given a triangle ABC with a set of concurrent cevians with feet D, E and F respectively on the sides BC, CA and AB, then starting from a point G, on side AB for example, and drawing GH//FE, HI//ED, IJ//DF, etc., produces a closed (crossed) hexagon GHIJKL."

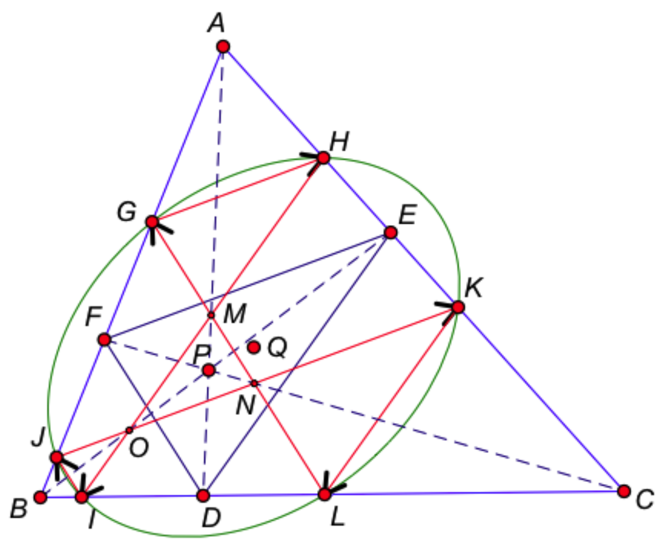

FIGURE 2

Before continuing, readers are invited to explore the result dynamically online at:
http://dynamicmathematicslearning.com/merry-go-round-triangle.html

Note that (i) G could be taken on AB produced (either way), and then some or all of the other points end up on sides produced, and (ii) P could be outside the triangle, or even at infinity (when the cevians are parallel).

[2] In a triangle, a cevian is a line segment connecting a vertex of the triangle to the opposite side (or its extension). The triangle formed by connecting the endpoints of the three cevians is known as the Cevian triangle. Medians, altitudes and angle bisectors are special cases of cevians. If the three cevians are concurrent, the point of concurrency is known as the Cevian point.

PROVING

By the parallel theorem mentioned earlier, we have $\frac{AF}{AE} = \frac{GF}{HE}$, $\frac{EC}{DC} = \frac{HE}{ID}$ and $\frac{BD}{BF} = \frac{ID}{JF}$. Multiplying and rearranging the three Left Hand Sides gives $\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA}$, and since AD, BE and CF are concurrent, by Ceva's theorem[3] the product must be 1. Multiplying the Right Hand Sides gives $\frac{GF}{JF}$ from which it follows that F is the mid-point of GJ. Now consider the other three parallels, from J to K, from K to L, and from L to G′ (not assuming this is G). By the same argument, F is the midpoint of G′J, and so we do indeed have G = G′.

Several further interesting properties are apparent in the configuration. For example, the intersection N of KJ and LG lies on CF. Likewise, the intersection M of LG and HI lies on AD, and the intersection O of HI and KJ lies on BE. This is easy to prove by means of a central dilation (enlargement) with fixed point C sending E to K and D to L: it must send EF to the parallel line KJ and likewise DF to LG. So F maps to the point of intersection of KJ and LG, so the point N must be on CF. Alternatively, we can use Desargues' theorem[4] as follows. The corresponding sides of triangles DEF and LKN are parallel, so they are in perspective from the line at infinity, and hence are in perspective from a point. But DL and EK meet at C, from which it follows that line FN must also pass through C. Likewise we can show that the points M and O respectively lie on AD and BE.

Since opposite sides of the crossed hexagon GHIJKL are parallel, they meet at infinity and these endpoints all lie on the line at infinity and hence, according to Pascal's theorem[5], the crossed hexagon lies on a conic. Varying G, while keeping the Cevian point P fixed, produces a pencil of conics, concentric and homothetic. When G, F and J are coincident the conic touches the sides of ABC at the vertices of the Cevian triangle, DEF.

According to a theorem related to hexagons with opposite sides parallel mentioned in (De Villiers, 2006), the three lines joining the midpoints of the opposite sides of GHIJKL meet at the common centre Q of these conics. A special case is when you take P as the Gergonne point[6] of ABC, when Q becomes the incentre and we get a set of concentric circles. There are obviously three other sets of concentric circles obtained somewhat similarly, and centred on the excentres. It is quite entertaining to work out for which P the conics are ellipses, or hyperbolae, or parabolas, or degenerate, and it is left to the more adventurous reader to investigate this further.

Lastly, in the spirit of the paper, readers are invited to formulate a possible converse, and explore whether or not it is valid. If valid, can readers supply a proof, or if false, produce a counter-example?

CONCLUDING REMARKS

Despite the current neglect in mathematics education of problem posing, Silver (1994) has proposed that problem posing is "central to the discipline of mathematics and the nature of mathematics thinking". According to him it is important for the following reasons:

(a) its relationship to creativity and exceptional mathematical ability,

(b) as a means of improving students' problem solving,

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[3] Ceva's theorem is stated on Wikipedia at: https://en.wikipedia.org/wiki/Ceva%27s_theorem

[4] The theorem of Desargues is stated at: https://en.wikipedia.org/wiki/Desargues%27s_theorem

[5] The theorem of Pascal is stated at: https://en.wikipedia.org/wiki/Pascal%27s_theorem

[6] The Gergonne point is defined at:
https://en.wikipedia.org/wiki/Incircle_and_excircles_of_a_triangle#Gergonne_triangle_and_point

(c) as a window into students' understanding of mathematics,

(d) as a way to improve students' disposition towards mathematics, and

(e) as a way to help students become autonomous learners.

The responsibility for formulating and solving new problems could rest as much with students as with their teachers, and students do not necessarily have to work on the problems they themselves pose, but could share their problems with each other and with their teacher (compare Silver, 1994 & Brown, 1984). In a similar vein, Schwartz (1992) has advocated the need for students to be taught to question, propose and explore conjectures, and to improve the learning of mathematics, "we must expand dramatically the time and attention we devote to the posing of problems."

Just as mathematics does not start with ready-made axioms, but with questions and problems, so a paper or lecture should not finish without a further question or problem. However, rather than posing further problems to the reader, it is hoped that readers here will have been stimulated to pose their own '*why not*' or '*what if*' questions, if not for this problem then for others they might be working on at the moment.

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