Proof of conjectured property of a cyclic quadrilateral

Let *ABCD* be a cyclic quadrilateral whose diagonals meet at *E*. The circumcentre and incentre of *ABE* are *F* and *G*, and of *CDE* are *H* and *I* respectively. Then

$$(DF^2 - CF^2) - (DG^2 - CG^2) = (AH^2 - BH^2) - (AI^2 - BI^2).$$

(*We have a cyclic quadrilateral, so the angle properties of circles might be useful, or the length properties from the theorem on intersecting chords. However, the second of these follows from the first, so let's concentrate on angle properties. These show at once that triangles ABE and CDE are similar.*
The differences of squares suggest factorization, but that is unusual in geometry. Squares of lengths occur in Pythagoras's theorem and its extension, the cosine rule. Perhaps there is a clue here.
We may have to use propeties of circumcentres and incentres, but at this stage it is not clear how.)

In triangles *ABE* and *DCE*, $\angle A = \angle D$, and $\angle B = \angle C$. Thus *ABE* and *CDE* are similar; their sides are therefore proportional. Let $DE/AE = \mu$, a constant. Then since *G* and *I*, and *F* and *H* are pairs of corresponding points in the two triangles, and *E* is common to both, we have $IE/GE = HE/FE = \mu$. Note too that *GEI* is a straight line, the angle bisector of $\angle AEB$. Also, by similarity, $\angle GEF = \angle IEF = \phi$, say.
Now, adding *AED* to each of the equal angles *AEG* and *DEI* gives $\angle GED = \angle AEI = \theta$, say. Similarly $\angle GEC = \angle BEI = \theta$.

(*If we think of each length that is squared as the side of a triangle whose third vertex is E, we can use the cosine rule to express them in terms of lengths each having E as an endpoint. In the subtractions some of these may cancel. The angles in the cosine rule will also be at E, and should be expressible using θ and ϕ.*)

Starting with the left-hand side of the conjecture we now have

$$DF^2 = DE^2 + EF^2 - 2\,DE\,.\,EF\cos(\theta+\phi)\,,\; CF^2 = CE^2 + EF^2 - 2\,CE\,.\,EF\cos(\theta-\phi)\,,$$
$$DG^2 = DE^2 + EG^2 - 2\,DE\,.\,EG\cos\theta\,,\; CG^2 = CE^2 + EG^2 - 2\,CE\,.\,EG\cos\theta\,.$$

When we substitute these into the expression, all the squared lengths cancel, and we are left with

$$2\,EF\left(CE\cos(\theta-\phi) - DE\cos(\theta+\phi)\right) + 2\,EG\left(DE - CE\right)\cos\theta\,.$$

The right-hand side gives

$$AH^2 = AE^2 + EH^2 - 2\,AE\,.\,EH\cos(\theta+\phi)\,,\; BH^2 = BE^2 + EH^2 - 2\,BE\,.\,EH\cos(\theta-\phi)\,,$$
$$AI^2 = AE^2 + EI^2 - 2\,AE\,.\,EI\cos\theta\,,\; BI^2 = BE^2 + EI^2 - 2\,BE\,.\,EI\cos\theta\,.$$

The right-hand side then simplifies to

$$2\,EH\left(BE\cos(\theta-\phi) - AE\cos(\theta+\phi)\right) + 2\,EI\left(AE - BE\right)\cos\theta\,.$$

But *EH* and *EF* are corresponding lines in the two triangles, with $EH = \mu\,EF$; similarly $EI = \mu\,EG$. We have also that $DE/AE = \mu$, i.e. $AE = DE/\mu$; similarly $BE = CE/\mu$. Making these substitutions in the right-hand side expression reduces it to that for the left-hand side. Hence the conjecture is true.

(*Since we have not used any properties of the circumcentres or incentres of the triangles, we can easily generalise the property. For instance F and H can be the orthocentres of the triangles, or the centroids, or the Fermat points, or even the midpoints, say, of corresponding medians, and so on. We can also replace G and I with the excentres opposite E, or with any other pair of corresponding points on the internal bisector of angle AEB.*)

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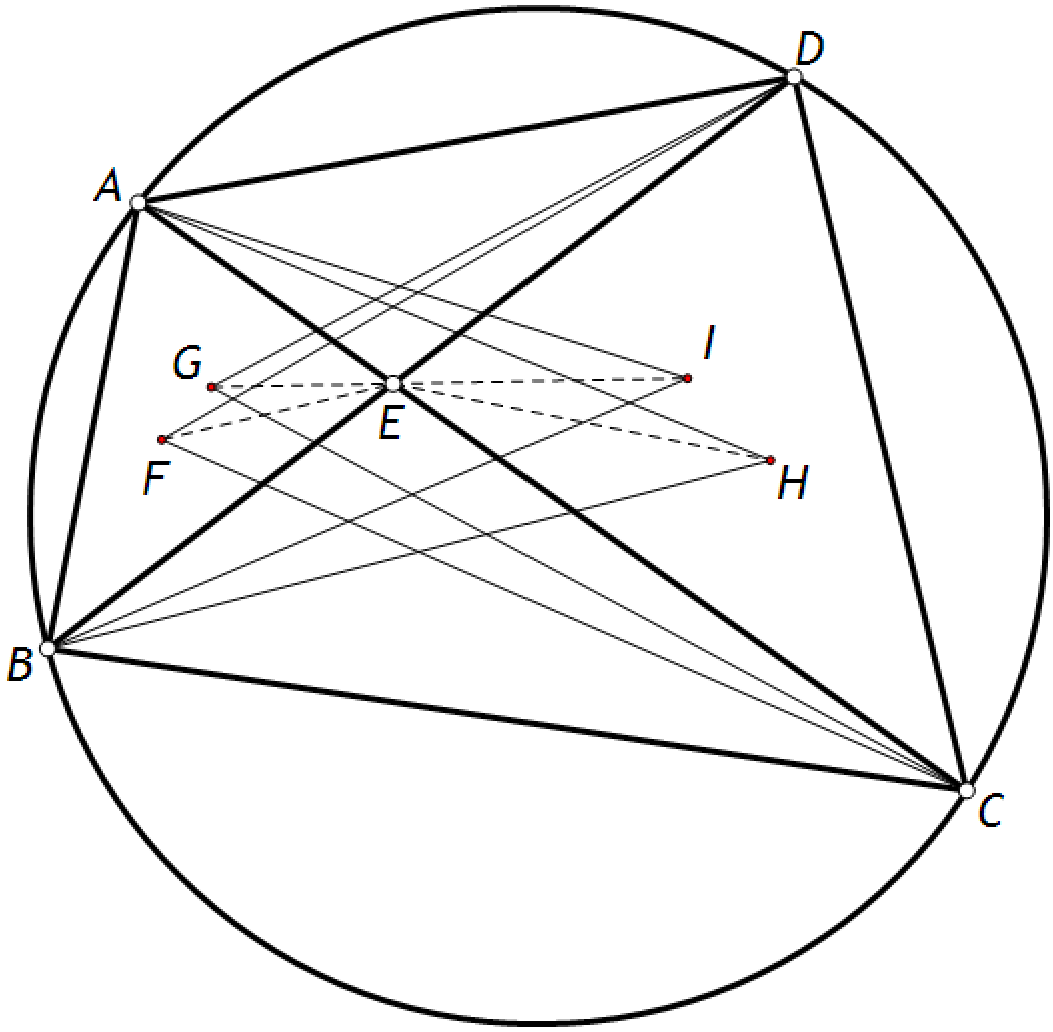
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