De Villiers, M. (2021). Published in The Mathematical Gazette, Volume 105 , Issue 564 , November, pp. 474 - 480.

Some more properties of the bisect-diagonal quadrilateral

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Martin Josefsson [1] has coined the term "bisect-diagonal quadrilateral" for a quadrilateral with at least one diagonal bisected by the other diagonal, and extensively explored some of its properties. This quadrilateral has also been called a "bisecting quadrilateral" [2], a "sloping-kite" or "sliding-kite" [3], or "slant kites" [4]. The purpose of this paper is to explore some more properties of this quadrilateral.

A familiar property of the bisect-diagonal quadrilateral that is proved in Coxeter [5, pp. 54-55) as well as in Josefsson [1, pp. 215], and is extended to the concave case by Pillay & Pillay [6, pp. 16-17] is the following:

Theorem 1: A quadrilateral is a bisect-diagonal quadrilateral (where at least one diagonal bisects the other) if, and only if, the diagonal that bisects the other also bisects the area of the quadrilateral.

Equi-partitioning point of a quadrilateral

As is well known, the centroid G of a triangle ABC divides, or equi-partitions, the triangle into three triangles, AGB, BGC, and CGA, of equal area.

The question now arises whether one can find a similar point P for a quadrilateral $ABCD$ that divides, or equi-partitions, it into four triangles, APB, BPC, CPD and DPA, of equal area. For a parallelogram, it's obvious that such an '*equi-partitioning*' point P exists, and would be located at its centroid, i.e. the intersection of its diagonals. But what about a more general quadrilateral? Where can P be located?

Based on the example of the triangle and the parallelogram, one may intuitively feel that in general such a point would be located at either the point mass centroid or the lamina centroid of a quadrilateral. However, a quick experimental check using an accurately constructed sketch with dynamic geometry as shown in Figure 1, shows that neither the point mass centroid[1] G_{PM} nor the lamina centroid[2] G_L respectively divide the quadrilaterals $ABCD$ and $KLMN$ into four triangles of equal area. Since G_L is the balancing point of the lamina (cardboard) quadrilateral $ABCD$, one would have anticipated that the four triangles subtended by G_L and the four sides would be equal in

[1] The point mass centroid of a quadrilateral is located at the intersection of the lines connecting the midpoints of opposite sides.

[2] The lamina centroid of a quadrilateral is located at the intersection of the line connecting the centroids of triangles KLM and MNK with the line connecting the centroids of triangles KLN and LMN.

area. This not being the case in general as shown in Figure 1, therefore seems a bit counter-intuitive and unexpected.

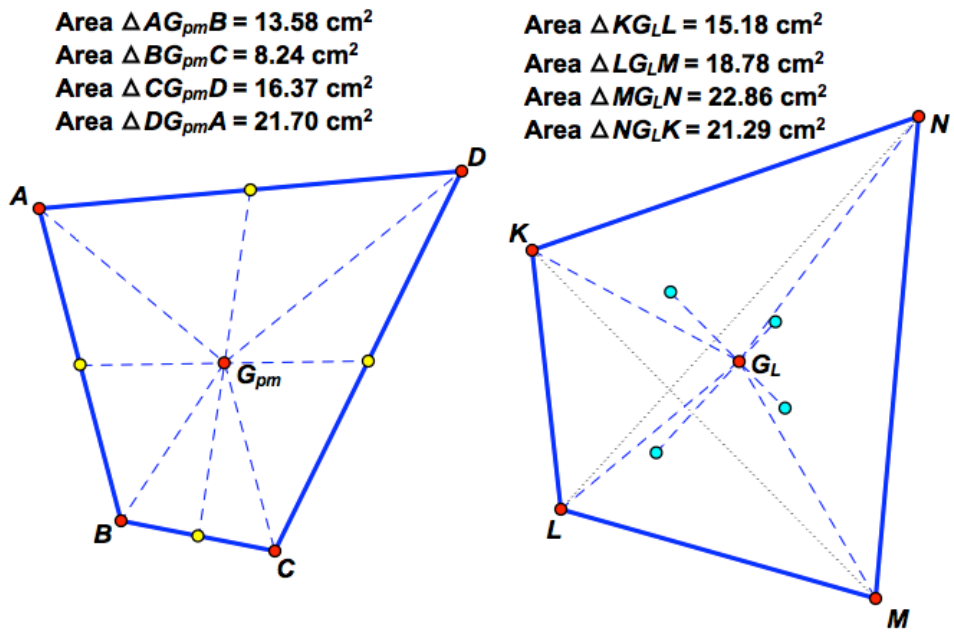

Figure 1

The reader may now wish to use the following online dynamic sketch to experimentally explore where such a point *P* might be located for a general quadrilateral or some special cases: http://dynamicmathematicslearning.com/equipartitioning-quad.html

Quite remarkably, such a (equi-partitioning) point *P* that divides, or equi-partitions, a quadrilateral into four triangles of equal area only exists for a bisect-diagonal quadrilateral. This follows from the following little known theorem proved by Pillay & Pillay [6] & Gilbert et al [7, pp. 68-70]:

Theorem 2: A quadrilateral has an equi-partitioning point *P* if, and only if, it is a bisect-diagonal quadrilateral, and then *P* is the midpoint of the diagonal bisecting the other.

The proof that the midpoint of the diagonal bisecting the other is the equi-partitioning point *P* of a bisect-diagonal quadrilateral follows directly from Theorem 1, and is left to the reader. The following proof that only a bisect-diagonal quadrilateral has an equi-partitioning point is slightly modified from that of [6] & [7], and is given below only for the convex case.

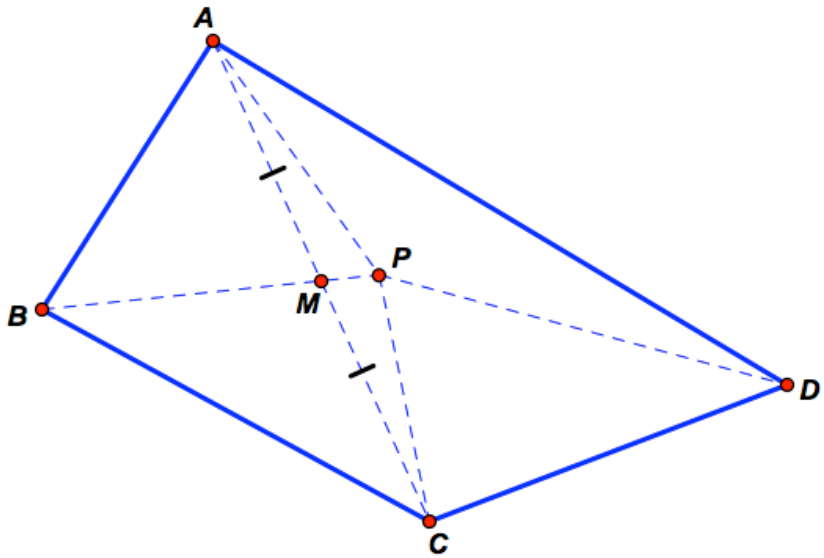

Figure 2

Proof: Suppose a convex quadrilateral *ABCD* has an equi-partitioning point *P* as shown in Figure 2. Since it is given that triangles *APB*, *BPC*, *CPD* and *DPA* are equal in area, it follows that diagonals *BP* and *DP* bisect the areas of quadrilaterals *ABCP* and *APCD* respectively. Hence from Theorem 1, both straight lines *BP* and *DP* extended contain the midpoint *M* of *AC*.

This implies that *DPM* is a straight line, and since the straight line through *M* and *P* must contain both *B* and *D*; we conclude that *BMPD* must coincide with the diagonal *BD*, and that *BD* bisects *AC* in *M*. But triangles *APB* and *DPA* have the same area, so *BP* = *PD*. Thus we have shown that diagonal *BD* bisects diagonal *AC* and that the equi-partitioning point *P* is the midpoint of *DB*.

Of course, the argument is entirely exchangeable, and we could in the same way argue that diagonal *AC* bisects diagonal *BD* and that the equi-partitioning point *P* is the midpoint of *AC*. Either way, the result is proved that at least one of the diagonals of *ABCD* is bisected by the other.

The same argument, with a few modifications, applies when quadrilateral *ABCD* is concave, but is left to the reader. As shown in [7, pp. 69-70], one can also prove this theorem using a trigonometric argument that extends to the concave case.

Lamina and point mass centroids of a bisect-diagonal quadrilateral

Let us now examine the lamina and point mass centroids of a bisect-diagonal quadrilateral, and any relationship between them.

Given a bisect-diagonal quadrilateral $ABCD$ as shown in Figure 4 with M the midpoint of the bisected diagonal BD and P the midpoint of diagonal AC. (According to Theorem 2, the point P is therefore the equi-partitioning point of $ABCD$).

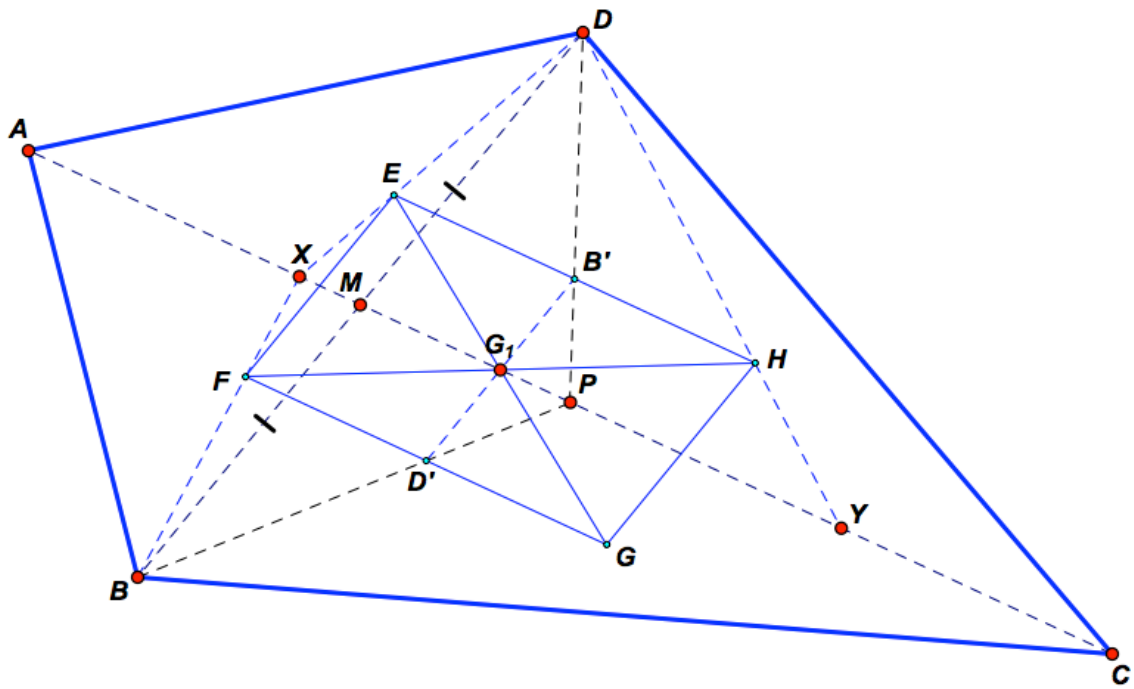

Figure 3

Theorem 3: Construct the centroids of triangles DPA, APB, BPC and CPD of a bisect-diagonal quadrilateral $ABCD$ and label them respectively, E, F, G and H. Then $EFGH$ is a parallelogram and the intersection of its diagonals, G_1, lies on AC, and is the lamina centroid of $ABCD$.

Proof: Since E lies on the median DX of triangle DPA and H lies on the median DY of triangle CPD, it follows that EH // XY and $EH = \frac{2}{3}XY$. Similarly, FG // XY and $FG = \frac{2}{3}XY$. Hence, opposite sides EH and FG are parallel and equal, and shows that $EFGH$ is a parallelogram. Since the areas of triangles DPA, APB, BPC and CPD are equal, the weight of their respective laminas would be equally concentrated at their centroids; hence all together, their lamina weights would balance at the intersection, G_1, of the diagonals of $EFGH$. More over, since EH and FG are the same distance away from AC, it follows that AC passes through the symmetrical point, G_1, of $EFGH$. This completes the proof of Theorem 3.

In addition, since $XP = PY$, note that B', the centroid of triangle ACD, is the midpoint of EH. Similarly, D' is the midpoint of FG. Since the centroids A' and C',

respectively, of triangles BCD and ABD, lie on diagonal AC, the line $B'D'$ also intersects the line $A'C'$ (line AC) at the lamina centroid, G_1.

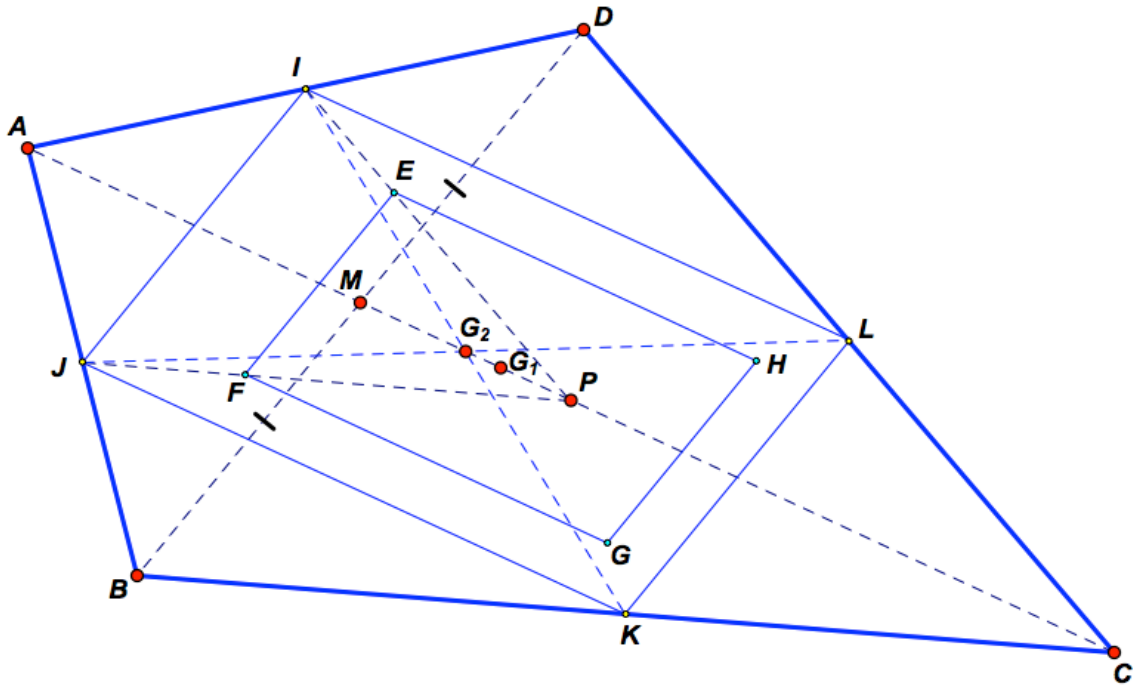

Figure 4

Theorem 4: The lamina parallelogram $EFGH$ of a bisect-diagonal quadrilateral $ABCD$ is homothetic to the Varignon parallelogram $IJKL$ formed by the midpoints of the sides of $ABCD$, with the centre of similarity between the two located at P, and a scale factor of $\frac{2}{3}$.

Proof: Since E and F are the respective centroids of triangles DPA and APB, we have in triangle IPJ that EF // IJ and $EF = \frac{2}{3}\, IJ$. Since the same can be shown for the other pairs of corresponding sides of $EFGH$ and $IJKL$, it follows that $EFGH$ is homothetic to $IJKL$ with centre P and scale factor $\frac{2}{3}$.

Theorem 5: The distance between the lamina centroid G_1 and the equi-partitioning point P of a bisect-diagonal quadrilateral is twice that of the distance between its lamina centroid G_1 and point mass centroid G_2.

Proof: Since the point mass centroid G_2 is located at the intersection of the diagonals of the Varignon parallelogram $IJKL$, it follows from the similarity transformation in Theorem 4 that $G_1P = 2G_2G_1$.

In addition, according to a well-known result in [5, p. 54] and [1, p. 216] the point mass centroid G_2 also lies at the midpoint of the line segment MP. Hence, $3G_2G_1 = G_2P$ $\Rightarrow 6G_2G_1 = MP$.

The Newton-Gauss line

Since the celebrated Newton–Gauss line [8, p. 62] is the straight line containing the midpoints of the three diagonals of a complete quadrilateral, it immediately follows that the diagonal AC passes through the midpoint S of the third diagonal QR of the complete bisect-diagonal quadrilateral $ABCDQR$.

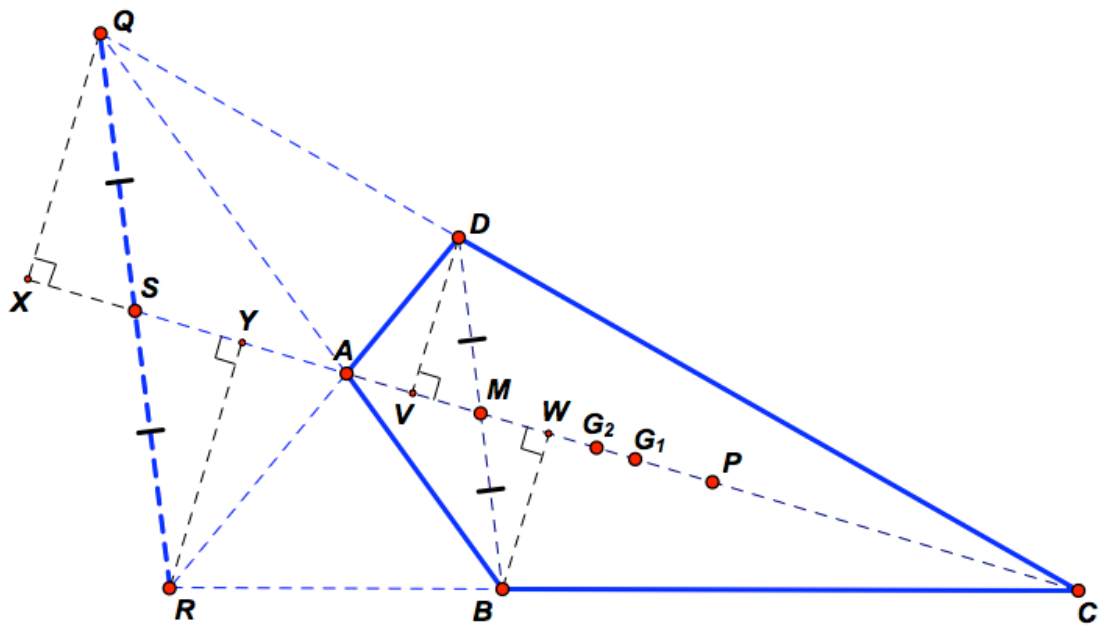

Figure 5

Theorem 6: Given a complete bisect-diagonal quadrilateral $ABCDQR$ as shown in Figure 5 with diagonal AC bisecting diagonal BD, then the third diagonal QR is parallel to BD.

Proof: Drop perpendiculars from Q, R, D and B to AC. From the similarity of triangles QXC and DVC it follows that $\frac{CD}{CQ} = \frac{DV}{QX}$. Similarly, $\frac{CB}{CR} = \frac{BW}{RY}$. From the congruency of triangles QXS and RYS, and of triangles DVM and BWM, we have $\frac{DV}{QX} = \frac{BW}{RY}$. Hence, $\frac{CD}{CQ} = \frac{CB}{CR}$, which implies that QR is parallel to BD.

Conversely, given a complete quadrilateral $ABCDQR$ with diagonal QR parallel to BD, then it's easy to see that the above argument applies in reverse, and that diagonal AC will bisect diagonal BD. In other words, $ABCD$ will be a bisect-diagonal quadrilateral.

Concluding comment

Apart from parallelograms and kites as special cases of a bisect-diagonal quadrilateral, it might also be of interest to some readers to note that that any cyclic quadrilateral *ABCD* with its sides $AB : BC : CD : DA$ in geometric progression with common ratio r, as shown in [9], is also a bisect-diagonal quadrilateral. It's easy to establish and left as an exercise.

Note: A dynamic geometry sketch illustrating the properties of a bisect-diagonal quadrilateral explored here is available online at:
http://dynamicmathematicslearning.com/bisect-diagonal-quadrilateral.html

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