Journal of Geometry
Vol. 33 (1988)

0047-2468/88/020129-18$1.50+0.20/0

NAPOLEON REVISITED

Dedicated to H. S. M. Coxeter on the occasion of his 80th birthday.

J. F. Rigby

Napoleon's Theorem can be neatly proved using a tessellation of the plane. The theorem can be generalized by using three similar triangles (instead of the three equilateral triangles) erected in different ways on the three sides of the triangle. Various interesting special cases occur.

1.

There is a well-known theorem attributed to Napoleon Bonaparte, although the authors of [4] doubt the possibility of his knowing enough geometry to prove the result [4, p.63]. The theorem can be stated as follows.

THEOREM 1.1. *If equilateral triangles are erected externally or internally on the sides of any triangle, their centres form an equilateral triangle.* (Figure 1A).

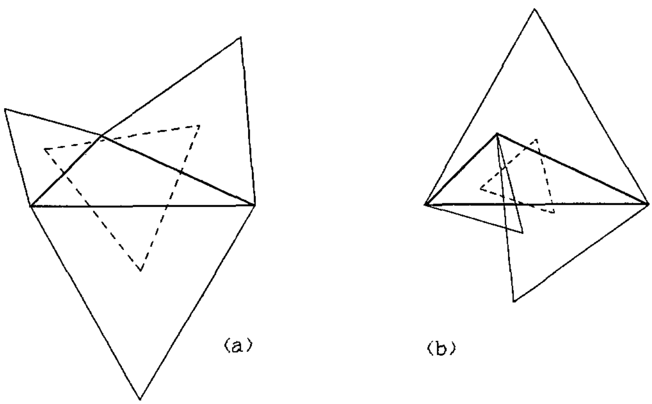

Figure 1A

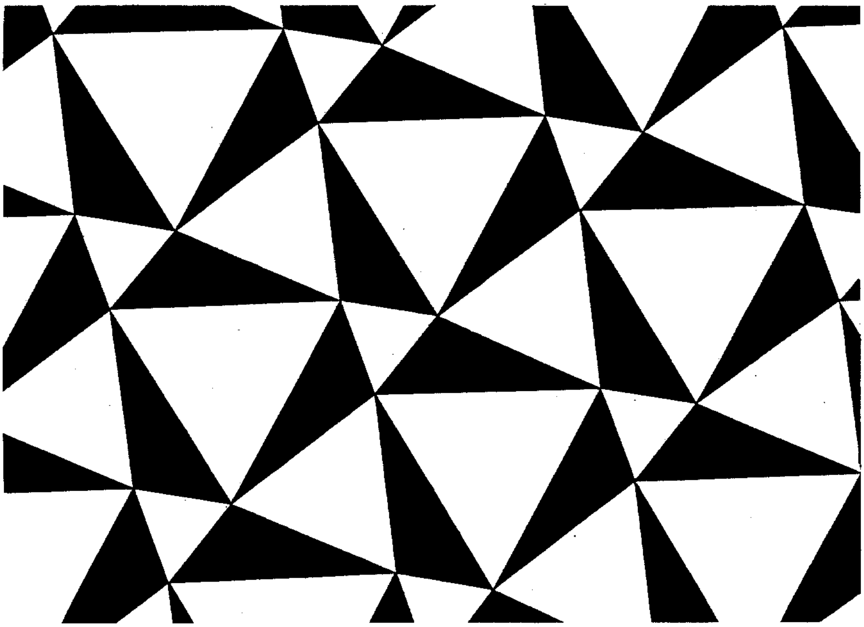

Figure 1B

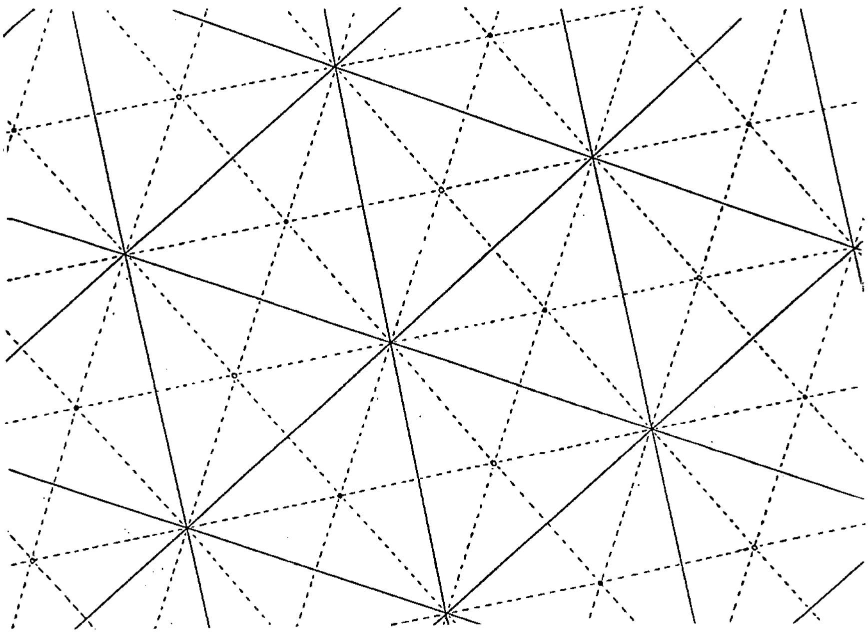

Figure 1C

My favourite method of seeing why this theorem is true is to extend Figure 1A(a) to a tessellation of the plane (Figure 1B). The centres of the small equilateral triangles clearly form an equilateral triangular lattice, and the centres of the other equilateral triangles of the tessellation lie at the centres of the triangles of the lattice. The lattice, and the centres of the other equilateral triangles, are shown in Figure 1C, from which it is clear that *all* the centres form a smaller equilateral triangular lattice; thus the truth of Napoleon's Theorem is demonstrated. This method works also if the equilateral triangles are erected internally, as long as we are prepared to extend our idea of a tessellation (Figure 1D).

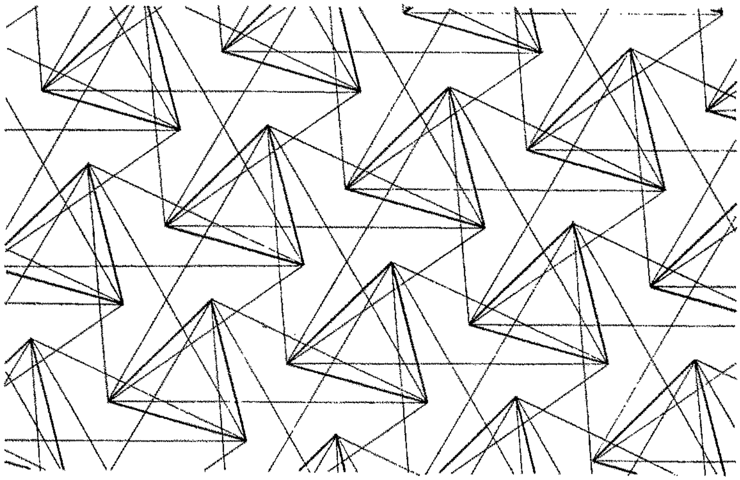

Figure 1D

A different method of proving Napoleon's Theorem leads to a generalisation that will be stated and proved in Section 3 (Theorem 3.1), but one special case of Theorem 3.1 will be mentioned now.

THEOREM 1.2. *If three similar triangles LBC, AMC, ABN are erected on the sides of any triangle ABC, their circumcentres form a triangle anti-similar to the three triangles.* (Figure 1E).

In [4, p.63] Napoleon's Theorem is obtained as a special case of the following result.

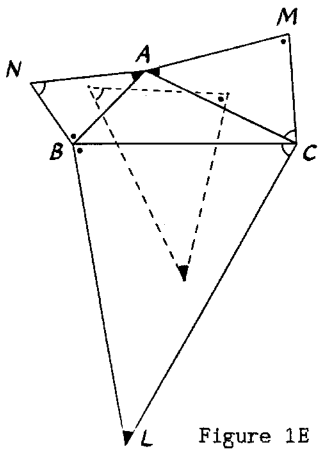

Figure 1E

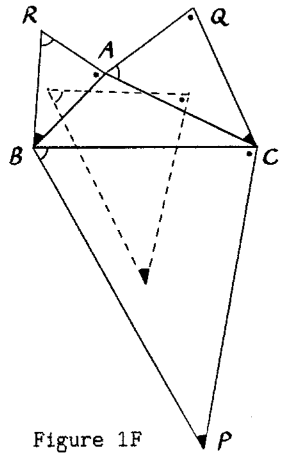

Figure 1F

THEOREM 1.3. *If three similar triangles PCB, CQA, BAR are erected externally on the sides of any triangle ABC, their circumcentres form a triangle similar to the three triangles* (Figure 1F).

When I first glanced at Theorem 1.3 in [4], I assumed that it was the same as Theorem 1.2, as I had not at that time distiguished between 'similar' and 'anti-similar' in the statement of Theorem 1.2, but a closer examination reveals that this is not so. It turns out that they are both special cases of two quite different theorems to be described in the next two sections (Theorems 2.5 and 3.1); although, if Figures 1D and 1E are superimposed, the two triangles formed by the circumcentres are seen to coincide.

It is worth giving a foretaste of the results in Section 2 by describing particular cases. Theorem 1.3 holds not just for the circumcentres of the similar triangles but for any points similarly situated in the triangles. For instance, the orthocentres of the triangles form a triangle similar to the three triangles, and so do the centroids, etc. Moreover, the circumcentre of the triangle formed by the orthocentres coincides with the orthocentre of the triangle formed by the circumcentres.

2.

We shall need to be careful about the measurement of angles. The angle CBA, denoted by <CBA (see Figure 1E for instance), is the angle between BC and BA; it is positive if the direction of rotation from BC to BA is anticlockwise, and

negative otherwise, so <CBA is a *signed* angle and <ABC = -<CBA. The *angles of the triangle ABC* are <BAC, <CBA and <ACB; these angles are positive if ABC is an anticlockwise triangle (i.e. if A, B, C occur in that order as we travel around the circumcircle in an anticlockwise direction) and they are negative if ABC is a clockwise triangle. If the angles of triangle ABC are denoted by a, b, c, then the angles of triangle ACB are -a, -c, -b.

Two triangles ABC and A'B'C' are *similar* if corresponding angles in the triangles are equal; they are *anti-similar* if the angles of triangle A'B'C' are the negatives of the corresponding angles of ABC. (The terms "directly similar" and "oppositely similar" are sometimes used instead.)

LEMMA 2.1. [3, p.75] *If AB and A'B' are two line segments, there is a unique dilative rotation (i.e. a dilatation followed by a rotation with the same centre), or in one special case a translation, mapping A,B to A',B'.*

As an immediate consequence of this lemma, we have

LEMMA 2.2. [3, p.75] *If triangles ABC and A'B'C' are similar, there is a unique dilative rotation, or in one special case a translation, mapping A,B,C to A',B',C'.*

We shall call the centre of the dilative rotation in Lemma 2.2 the *dilative centre* of ABC and A'B'C'.

The next lemma is obvious from Figure 2A, using similar triangles.

LEMMA 2.3. *If there is a dilative rotation with centre O, or a translation, mapping A,B to C,D, then there is a dilative rotation with centre O, or a translation, mapping A,C to B,D.*

We shall abbreviate the statement of this lemma in the following form:

$$\text{if } A,B \to C,D \text{ then } A,C \to B,D.$$

Since all the dilative rotations in this section will have the same centre, no confusion will be caused by this short notation.

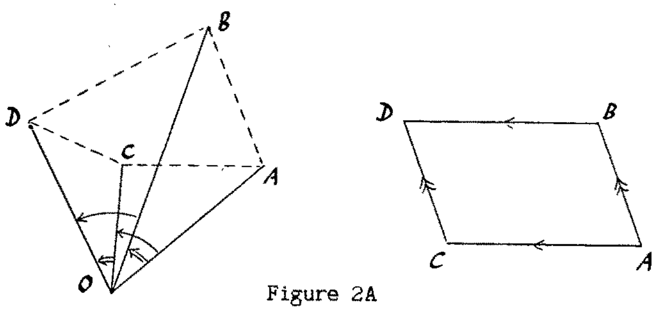

Figure 2A

Let ABC and A'B'C' be similar triangles; let X be any point, and regard X as being "a point of the triangle ABC", even though it may lie inside, on, or outside the triangle. Suppose that the dilative rotation mapping ABC to A'B'C' also maps X to X'; then X and X' are *corresponding points* of ABC and A'B'C'. Examples of corresponding points are the circumcentres of the triangles, or their orthocentres, centroids, incentres, or any other well-known points associated with the triangles.

THEOREM 2.4. *If similar triangles PCB, CQA, BAR are erected on the sides of any triangle ABC, then the dilative centres of the triangles taken in pairs all coincide, unless the triangles are congruent with parallel sides as in Figure 2B.*

Proof. (Figure 2C) Disregarding the special case, let O be the dilative centre of PCB and CQA. Then C,B → Q,A. Hence C,Q → B,A by Lemma 2.3. Hence C,Q,A → B,A,R as required, by the uniqueness in Lemma 2.1.

THEOREM 2.5. *If three similar triangles PCB, CQA, BAR are erected on the sides of any triangle ABC, and if U, V, W are corresponding points of the three triangles, then UVW is similar to the three triangles.*

Proof. (Figure 2C) We have Q,V → C,U and Q,V → A,W; hence by Lemma 2.3 Q,C → V,U and Q,A → V,W. But there is a unique dilative rotation with centre O mapping Q to V; hence Q,C,A → V,U,W.

THEOREM 2.6. *In the situation of Theorem 2.5, let U', V', W' and T' be corresponding points in the triangles PCB, CQA, BAR and UVW. Then U, V, W and T' are corresponding points in the triangles PCB, CQA, BAR and U'V'W'.* (Figure 2C).

Proof. We have C,V' → U,T'; hence C,U → V',T' by Lemma 2.3. Also P,C,B → U',V',W' by Theorem 2.5. But there is a unique dilative rotation with centre O mapping C to V'; hence P,C,B,U → U',V',W',T'.

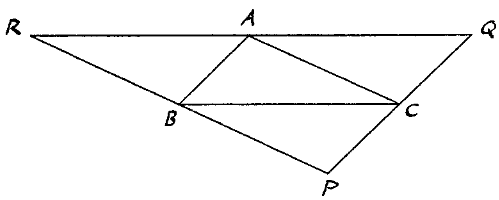

Figure 2B

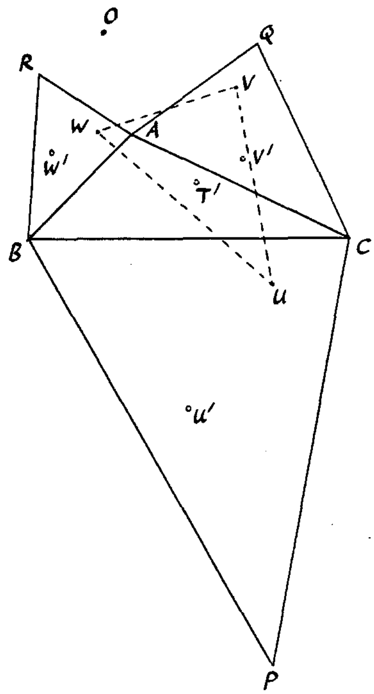

Figure 2C

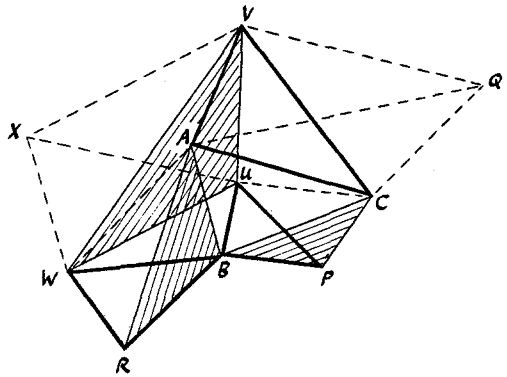

Figure 2D

In the situation of Theorems 2.5 and 2.6, let us call U the *λ-point* of triangle PCB, and U' the *μ-point* of triangle PCB. Then V and V' are the λ-point and the μ-point of triangle CQA, etc., and we can call UVW and U'V'W' the λ-triangle and the μ-triangle. Then Theorem 2.6 can be neatly expressed in the following form.

The μ-point of the λ-triangle coincides with the λ-point of the μ-triangle.

Chris Fisher has pointed out that Theorem 2.5 appears implicitly in a paper by B. H. Neumann in 1941 [5, p.238]. Neumann's figure, labelled according to our present notation, is shown by the unbroken lines in Figure 2D, and his version of Theorem 2.5 is: *if the triangles PBU, CAV, BRW are similar, and if PCB is similar to BAR, then these last two are also similar to UVW.* In this figure PCBU, CQAV, BARW are congruent quadrangles; if X is the point such that UVWX is congruent to the other three quadrangles, the figure illustrates Theorem 2.5 in four different ways: we can use UWB, VUC or WVA as the "initial triangle" in place of ABC.

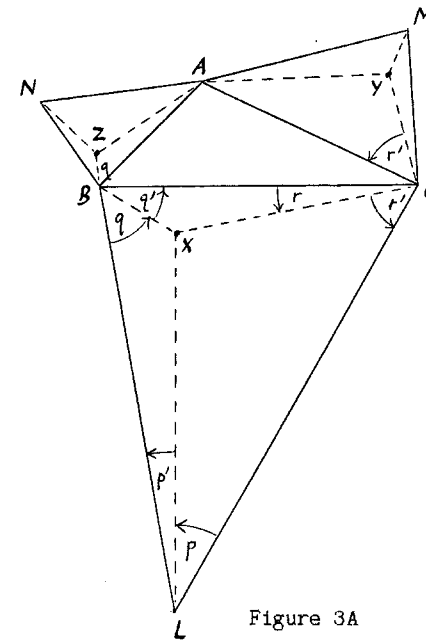

Figure 3A

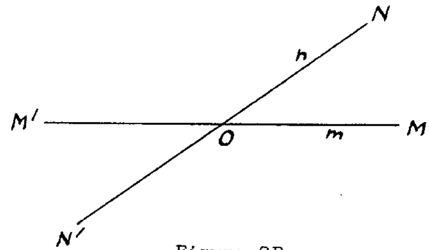

Figure 3B

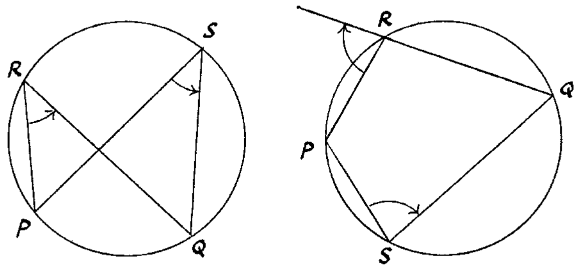

Figure 3C

3.

THEOREM 3.1. *Let three similar triangles LBC, AMC, ABN be erected on the sides of a triangle ABC, and let X, Y, Z be corresponding points of the three triangles with angles as shown in Figure 3A. Then the angles of triangle XYZ are* $q + r'$*,* $r + p'$*,* $p + q'$*; so the shape of triangle XYZ depends on the shape of LBC and on the position of X, but is independent of the shape of ABC.*

Proof. Let γ denote the dilative rotation with centre C and scale factor CL/CX through the angle r', and let β denote the dilative rotation with centre B and scale factor BX/BL through the angle q. Then

$$X\gamma\beta = L\beta = X \text{ and } Y\gamma\beta = A\beta = Z.$$

Hence X is the centre of the dilative rotation $\gamma\beta$, and since the angle of this dilative rotation is $q + r'$ it follows that $\angle YXZ = q + r'$. Similarly $\angle ZYX = r + p'$ and $\angle XZY = p + q'$.

Proof of Theorem 1.2. If X is the circumcentre of triangle LBC, then (with the notation of Figure 3A) $r' = p$, $p' = q$, $q' = r$. Hence $\angle YXZ = p + p' = \angle CLB = -\angle BLC$. Similarly $\angle ZYX = q + q' = \angle LBC = -\angle CBL$ and $\angle XZY = r + r' = \angle BCL = -\angle LCB$. Hence triangle XYZ is anti-similar to triangle LBC.

A further word about the measurement of angles is appropriate here. In the subsequent proofs it will be convenient to measure the angle between two lines modulo 180°, taking the sign into account. In Figure 3B for instance, the angle between the lines m and n is 35° or -145°, and we write $<mn \equiv 35° \equiv -145°$, whereas $<nm \equiv -35°$. In the same figure we can also denote $<mn$ by $<MON$, $<M'ON$, $<MON'$ or $<M'ON'$. It will usually be convenient to use the symbol '=' rather than '≡'. Two well-known circle theorems (Euclid III, 21 and 22), with their converses, can now be combined in the following concise form: *The four points P, Q, R, S are concyclic if and only if* $<PRQ = <PSQ$ (Figure 3C), irrespective of the order of the points on the circle. Note however that in the proof of Theorem 3.1 the angles must be measured modulo 360°: a rotation through 180° is not the same as the identity.

LEMMA 3.2. *Given three similar triangles LBC, AMC and ABM on the sides of triangle ABC, and a further triangle FGH that is not (directly) similar to the first three, we can choose the points X, Y, Z in Theorem 3.1 uniquely in such a way that XYZ is similar to FGH.*

Proof. Denote the angles of triangle XYZ in Figure 3A by x, y, z and the angles of triangle LBC by l, m, n; then $x = q + r'$ etc. by Theorem 3.1, and $l = -(p + p')$ etc. Denote the angles of triangle FGH by f, g, h. Then

$$<CXB = r' + p + p' + q = x - l;$$

hence $x = f$ if and only if $<CXB = f - l$; this condition is satisfied if and only if X lies on a particular circle through B and C. Similarly $y = g$ and $z = h$ if and only if $<LXC = <AYC = g - m$ and $<BXL = <BZA = h - n$. Since

$$(f - l) + (g - m) + (h - n) \equiv 0°,$$

all three conditions are satisfied if and only if X is the second intersection of two particular circles, through B,C and through C,L. This argument breaks down if $f - l = 0°$ and $g - m = 0°$ (i.e. if FGH is similar to LBC), for then the two circles become the lines BC and CL.

LEMMA 3.3. *Let X',Y',Z' and X",Y",Z" be two sets of corresponding points in Theorem 3.1. Then the triangles X'Y'Z' and X"Y"Z" are anti-similar if and only if X' and X" are inverse points in the circumcircle of triangle LBC.*

Proof. Suppose that X'Y'Z' and X"Y"Z" are anti-similar, and denote the angles of X'Y'Z' by x, y, z. Then, as in the proof of Lemma 3.2, $<CX'B = x - l$ whilst

<CLB = −l. Hence the angle at B between the circles BCL and BCX' is x (Figure 3D). Similarly the angle at B between the circles BCL and BCX" is −x, since the angles of triangle X"Y"Z" are −x, −y, −z. Hence, because of the angle-reversing property of inversion, circle BCX" is the inverse of BCX'. Similarly CLX" is the inverse of CLX'. It follows that X" is the inverse of X'. The converse is proved similarly.

Now if X'Y'Z' is to be anti-similar to LBC, X' must be the circumcentre of LBC, by Theorem 1.2 and by the uniqueness in Lemma 3.2. Hence if X"Y"Z" were to be similar to LBC, X" would have to be the inverse of the circumcentre, namely the point at infinity. This is another way of looking at the exceptional case in Lemma 3.2.

An investigation of the question "When is XYZ an equilateral triangle?" leads to the following definition of the *Napoleon points* of a triangle. Let LBC be a triangle (Figure 3E) with angles l, m, n. There exists a point D such that

<CDB = l + 60° and <LDC = m + 60°:

D is uniquely determined as the second point of intersection of a circle through B,C and a circle through C,L. It follows that <BDL = n + 60°. Similarly there exists a point E such that

<CEB = l − 60°, <LEC = m − 60°, <BEL = n − 60°.

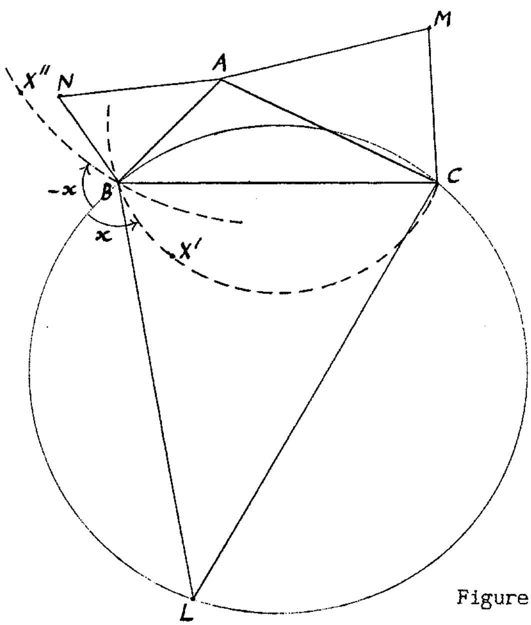

Figure 3D

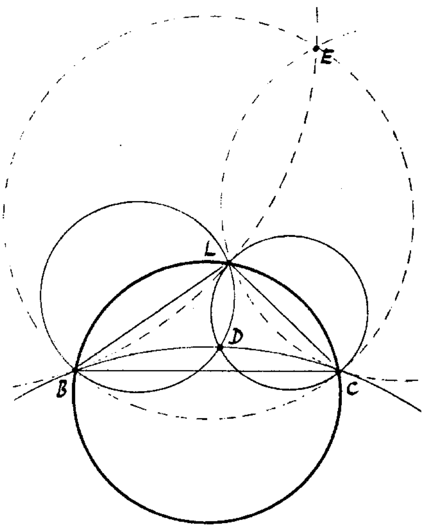

Figure 3E

There may well be an existing name for D and E, but we shall call them the *Napoleon points* of triangle LBC. They are inverse points in the circumcircle of LBC (this can be shown by the method of proof used in Theorem 3.3). If LBC is equilateral then one Napoleon point is at the centre of LBC whilst the other does not exist (or lies at infinity).

The circles used to determine D and E (Figure 3E) intersect the circle LBC at angles of 60°. Hence if we invert Figure 3E we obtain another triangle with its Napoleon points. The author knows of no other interesting points connected with a triangle that are transformed by every inversion to the corresponding points of the inverse triangle.

THEOREM 3.4. *With the notation of Lemma 3.3, if X' is one of the Napoleon points of triangle LBC then triangle X'Y'Z' is equilateral, whilst if X" is the other Napoleon point then X"Y"Z" is equilateral but described in the opposite sense.*

Proof. This follows from the proof of Lemma 3.2.

4.

In this section we consider further special positions of the points X,Y,Z in Theorem 3.1.

THEOREM 4.1. *If X lies on the circumcircle of LBC then X,Y,Z are collinear.*

Proof. Using the notation of Theorem 3.1, we easily see that when X lies on the circumcircle then $q + r' = r + p' = p + q' = 0°$. Alternatively, using the notation of Lemma 3.3, if X' lies on the circumcircle then X' coincides with the inverse point X"; the triangles X'Y'Z' and X"Y"Z" are anti-similar and yet they coincide, and this can only happen if X',Y',Z' are collinear.

THEOREM 4.2. *(a) The circumcircles of triangles LBC, AMC, ABN are concurrent at a point O.*

(b) The point O is also the meet of the lines AL, BM, CN.

(c) The line XYZ in Theorem 4.1 passes through O.

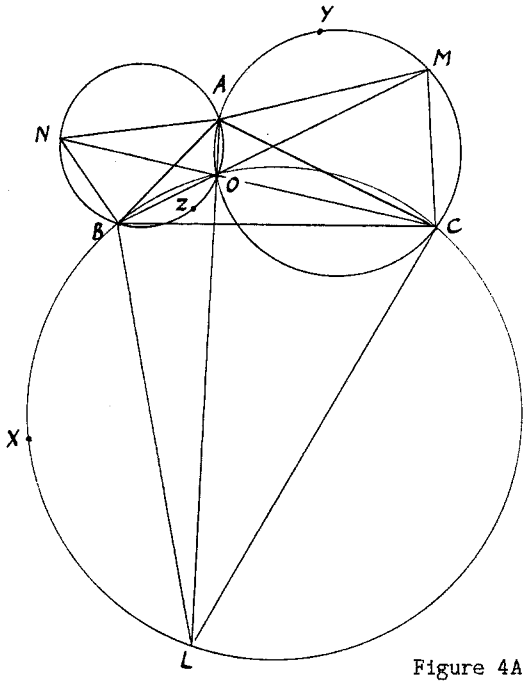

Figure 4A

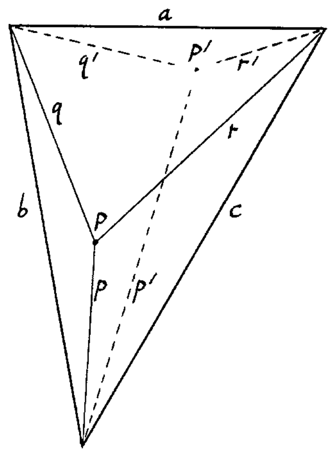

Figure 4B

Proof. (Figure 4A) (a) Let the circles LBC, AMC meet at O. Then, using the same notation as in the proof of Lemma 3.2, <COB ≡ <CLB = -1 and <AOC ≡ <AMC = -m. Hence <BOA = 360° - <COB - <AOC ≡ 1 + m ≡ -n = <BNA. Hence O lies on the circle ABN.

(b) <LOB ≡ <LCB = n = <ANB ≡ <AOB; hence A,O,L are collinear, so O lies on AL. Similarly O lies on BM and CN.

(c) Let X lie on the circle LBC and let Y be the corresponding point on the circle AMC. Then <XOL ≡ <XCL = <YCA ≡ <YOA ≡ <YOL; hence X,Y,O are collinear. Similarly Y,Z,O are collinear. The result follows, and we have also another proof of Theorem 4.1.

When X is at L, then Y and Z coincide at A; hence AL, and similarly BM and CN, are particular positions of XYZ.

For a given triangle ABC, we can choose X,Y,Z in Theorem 3.1 so that the triangle XYZ is similar to ABC. This does not contradict the statement in Theorem 3.1 that the shape of XYZ is independent of the shape of ABC, because here we choose X *after ABC is given*. Lemma 3.2 tells us that we cannot make this choice of X if

ABC is similar to LBC; but this happens only when L = A, M = B and N = C, and we can ignore this trivial case.

THEOREM 4.3. *If X is chosen so that XYZ is similar to ABC, then XYZ is congruent to ABC and is obtained by rotating ABC through 180°.*

Proof. In this proof we need the concept of *isogonal conjugates,* which is illustrated in Figure 4B. If the lines p,q,r through the vertices of a triangle (with sides a,b,c) are concurrent in P, and if p', q', r' are lines through the vertices such that <cp = <p'b etc. as shown, then p', q', r' are concurrent in a point P' called the *isogonal conjugate* of P with respect to the triangle ABC [see 4, p.93, 2, p.49, 1, p.16 for various proofs].

Let the lines AOL, BOM, CON in Figure 4A meet the circle ABC again in F,G,H as in Figure 4C, and label the angles <CBO etc. in triangle ABC as shown. Then <CLO = <CBO = 1 and <OLB = <OCB = 2. Similarly <BNA = 5 + 6; hence <BCL = 5 + 6. But <BCF = <BAF = 5, so <FCL = 6. In this way we can label all the angles in the figure as shown. Now let O', F', G', H' be the isogonal conjugates of O, F, G, H with respect to the triangles ABC, LBC, AMC, ABN respectively; we obtain Figure 4D with the angles as shown. From various paralellograms in the figure we see (using

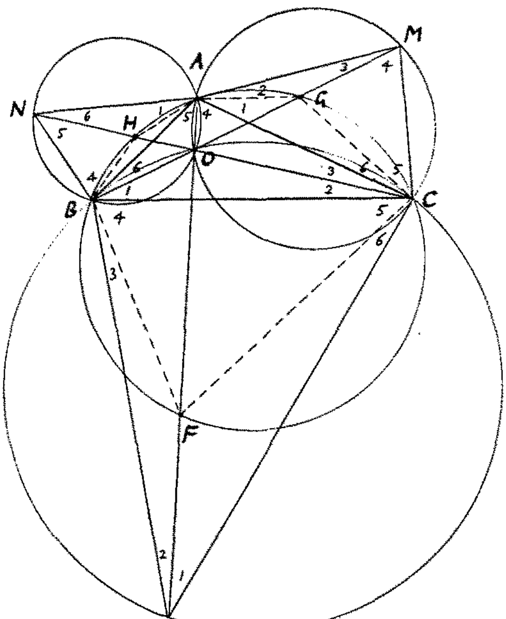

Figure 4C

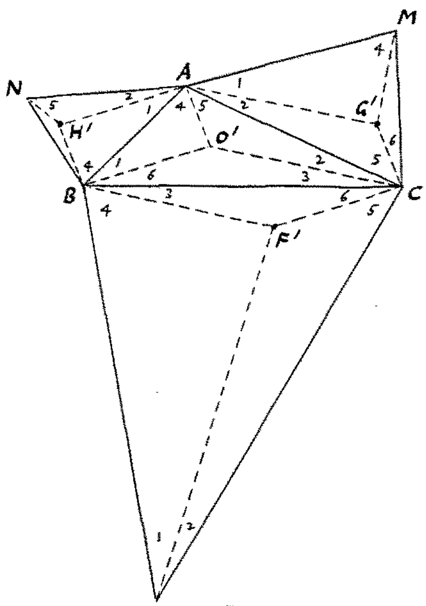

Figure 4D

vector notation) that $\overrightarrow{BH'} = \overrightarrow{O'A} = \overrightarrow{CG'}$, so $\overrightarrow{G'H'} = -\overrightarrow{BC}$. Similarly $\overrightarrow{H'F'} = -\overrightarrow{CA}$ and $\overrightarrow{F'G'} = -\overrightarrow{AB}$. The result now follows because F', G', H' are corresponding points of the triangles LBC, AMC, ABN, so that F', G', H' are the points X, Y, Z.

5.

Let us now return to the special case where the triangles LBC, AMC and ABN are equilateral. In Figure 5A, X_1 is any point associated with triangle LBC, and X_2, ... X_6 are the images of X_1 under the various symmetries of the triangle. The points Y_1, ..., Y_6 and Z_1, ..., Z_6 are the points of triangles AMC and ABN corresponding to X_1, ..., X_6.

THEOREM 5.1. *Let FGH denote any triangle similar to triangle $X_1Y_1Z_1$ in Figure 5A. Then $X_3Y_3Z_3$ and $X_5Y_5Z_5$ are similar to GHF and HFG respectively, whilst $X_2Y_2Z_2$, $X_4Y_4Z_4$ and $X_6Y_6Z_6$ are anti-similar to FHG, HGF and GFH respectively. This means that $X_1Y_1Z_1$, $Z_3X_3Y_3$ and $Y_5Z_5X_5$ are similar, whilst $X_2Z_2Y_2$, $Z_4Y_4X_4$ and $Y_6X_6Z_6$ are anti-similar to $X_1Y_1Z_1$.*

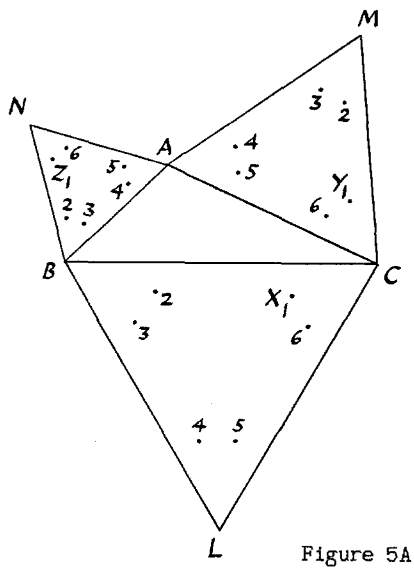

Figure 5A

Proof. This follows immediately from Theorem 3.1 when we investigate the angles in the figure.

THEOREM 5.2. *The triangles* $X_1Y_1Z_1$, $Z_3X_3Y_3$ *and* $Y_5Z_5X_5$ *are congruent and are oriented at angles of 120° to each other, whilst* $X_2Z_2Y_2$, $Z_4Y_4X_4$ *and* $Y_6X_6Z_6$ *are all anti-congruent (i.e. oppositely congruent) to* $X_1Y_1Z_1$ *and are oriented at angles of 120° to each other.*

Proof. The angles $\langle X_1CX_6$ and $\langle Y_1CY_6$ are equal. Hence there is a rotation about C mapping X_1, Y_1 to X_6, Y_6. Hence $X_1Y_1 = X_6Y_6$. Hence triangles $X_1Y_1Z_1$ and $Y_6X_6Z_6$ are anti-congruent, for we already know that they are anti-similar. Similarly $Z_3X_3Y_3$ and $Z_4Y_4X_4$, and also $Y_5Z_5X_5$ and $X_2Z_2Y_2$, are anti-congruent. By considering suitable rotations about B also, we show that all the triangles are congruent or anti-congruent.
The points X_1, Y_3 and Z_5 are corresponding points of the triangles LCB, CMA and BAN. Hence by Theorem 2.5 the triangle $X_1Y_3Z_5$ is equilateral. Also $Z_1Z_3Z_5$ is equilateral. Hence the rotation about Z_5 through an angle of 60° maps X_1Z_1 to Y_3Z_3. Hence there is a rotation through an angle of -120° mapping X_1Z_1 to Z_3Y_3; this rotation must map triangle $X_1Y_1Z_1$ to the congruent triangle $Z_3X_3Y_3$. The rest of the theorem is proved in a similar way.

If we choose X_1 so that $X_1Y_1Z_1$ is congruent to ABC (see Theorem 4.3) then $X_2Z_2Y_2$, $Z_4Y_4X_4$ and $Y_6X_6Z_6$ are all anti-congruent to ABC, so we can certainly obtain these triangles from ABC by applying glide reflections. In fact, the glide reflections turn out to be simple reflections, as stated in the theorem below; we leave the proof to the reader.

THEOREM 5.3. *If we choose* X_1 *in Theorems 5.1 and 5.2 so that the triangle* $X_1Y_1Z_1$ *is congruent to ABC, then* $X_2Z_2Y_2$, $Z_4Y_4X_4$ *and* $Y_6X_6Z_6$ *can be obtained by reflecting ABC in three lines making angles of 60° with each other.*

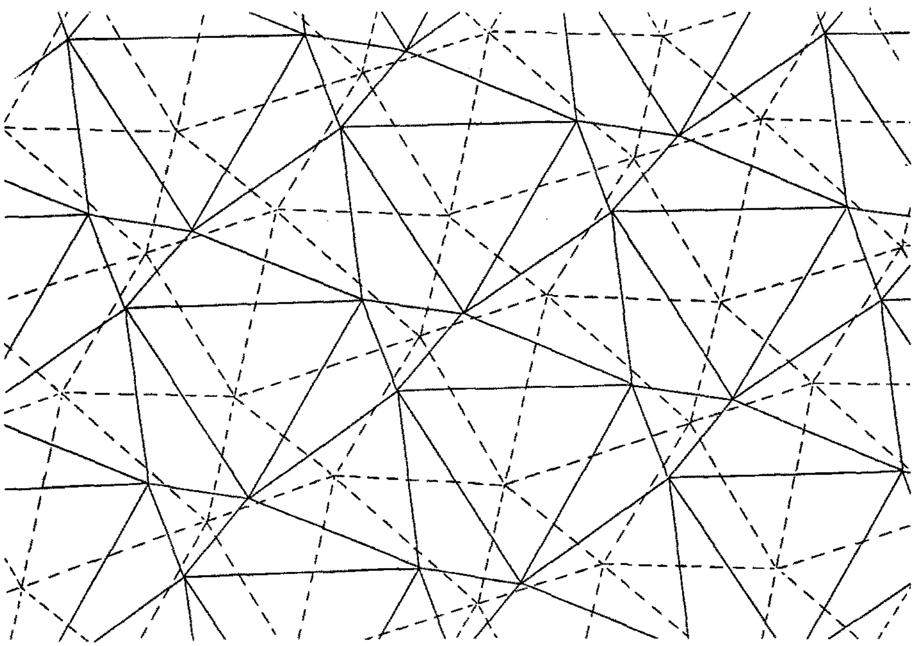

Figure 6A

6.

We return finally to the tessellation of Section 1. There appears to be no way of using either Figure 2C or Figure 3A to form a tessellation unless triangle PCB or triangle LBC is equilateral. If we take the triangles of Figure 5A, with corresponding points X, Y, Z in the triangles LBC, AMC, ABN, and embed them repeatedly (without rotation) in the tessellation of Figure 1B, then the points X, Y, Z and all their repetitions form another tessellation of the same type as the original one, but with different angles in general (Figure 6A). This follows from Theorem 5.2. With a little more investigation of angles we see that the original tessellation can be obtained from the new one in a similar manner.

If X_1 is chosen in Figure 5A so that $X_1Y_1Z_1$ is similar to ABC, and if X is then chosen to be one of the points X_1, ..., X_6 in Figure 5A, the new tessellation is a rotation or a reflection of the original. This follows from Theorems 4.3, 5.2 and 5.3; the details are left to the reader.

Note that although the separate tessellations in Figure 6A have threefold rotational symmetry, this is not true of the two tessellations together.

If X is now taken to lie on the circumcircle of triangle LBC, then Y and Z lie on the circumcircles of AMC and ABN, and X, Y, Z are collinear (Theorems 4.1, 4.2) so the triangle XYZ is degenerate. When we embed this figure in the tessellation we obtain the heavy lines of Figure 6B; the original tessellation has not been drawn here, but the circumcircles of the original equilateral triangles are shown. The figure is an attempt to illustrate what happens when X moves around the circumcircle; further visual insight could be obtained from a moving film of the situation. (Note that the original triangle ABC on which the tessellation of Figure 6B is based is not the same triangle as in Figure 6A.)

I am grateful to the referee for suggesting various improvements.

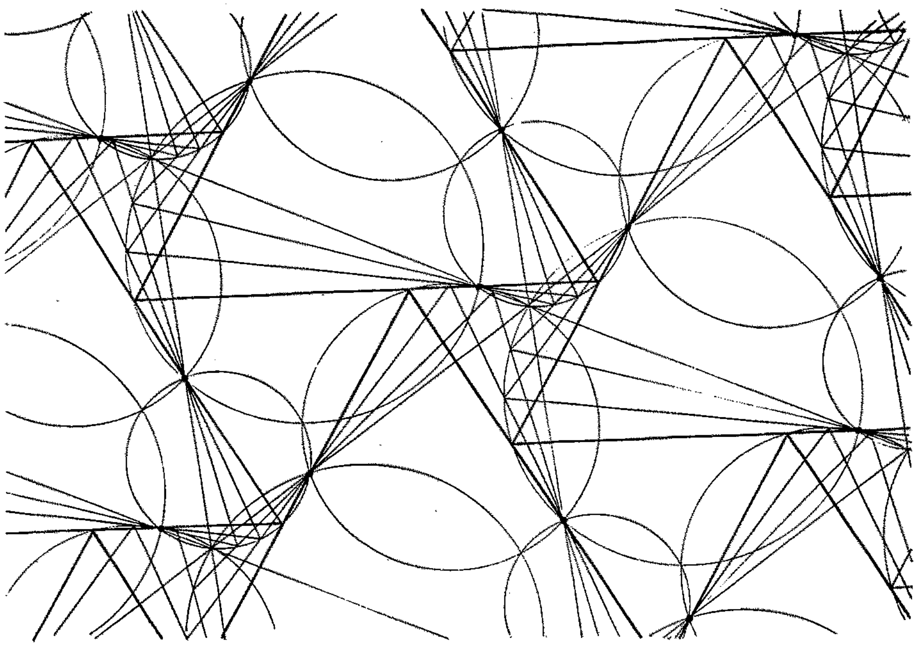

Figure 6B

REFERENCES

[1] BACHMANN, F.: *Aufbau der Geometrie aus dem Spiegelungsbegriff*, Springer, Berlin 1959, 1973.

[2] COOLIDGE, J. L.: *A Treatise on the Circle and the Sphere*, Oxford University Press 1916.

[3] COXETER, H. S. M.: *Introduction to Geometry*, 2nd ed., Wiley, New York 1969.

[4] COXETER, H. S. M. and GREITZER, S. L.: *Geometry Revisited*, Mathematical Association of America, Washington DC, 1967.

[5] NEUMANN, B. H.: *Some remarks on polygons*, J. London Math. Soc. *16* (1941), 230-245.

Department of Pure Mathematics
University College
P. O. Box 78
Cardiff CF1 1XL
WALES U.K.

(Eingegangen am 22. Oktober 1986)