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Napoleon’s Theorem with Weights in n-Space

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Abstract. The famous theorem of Napoleon was recently extended to higher dimensions. With the help of weighted vertices of an n-simplex T in $\mathbb{E}^n$, $n \geqslant 2$, we present a weighted version of this generalized theorem, leading to a natural configuration of $(n-1)$-spheres corresponding with T by an almost arbitrarily chosen point. Besides the Euclidean point of view, also affine aspects of the theorem become clear, and in addition a critical discussion on the role of the Fermat–Torricelli point within this framework is given.

0. Introduction

It is known that Napoleon Bonaparte was interested in mathematics and natural sciences, but it is not completely verified whether the basic theorem discussed here is due to him. Its first attribution seems to have been given in the book [Fa], where it is accompanied by the paranthetical comment ‘Teorema proposto per la dimonstrazione da Napoleone a Lagrange’ (p. 186). However, the theorem itself occurred already in the Italian school book [Tu] from 1843, see also [La].

In the recent survey [Ma] many older and new contributions to Napoleon’s theorem, its generalizations and relatives are discussed. But the extensive list of references in [Ma] shows that until now spatial analogues to the theorem were (almost) not considered. This led B. Weißbach [We] to extend a restricted version of the theorem to Euclidean n-space, $n \geqslant 3$. Here we will show that the configuration described by Napoleon’s theorem can be considered as a special case of a configuration which consists of an n-simplex T and certain $(n-1)$-spheres associated with T by an almost arbitrarily chosen point. This yields the weighted version of Napoleon’s theorem for $n \geqslant 3$, and besides the metrical aspects of this configuration also its affine properties become clear. In addition, a critical view on the role of the Fermat–Torricelli point in this connection is presented.

1. The Configuration of Torricelli’s

Our starting point is Napoleon’s theorem itself. Let $p_0p_1p_2 \subset \mathbb{E}^2$ denote an arbitrary triangle with vertices p_0, p_1, p_2, and let $p_0^*p_1p_2$, $p_0p_1^*p_2$, $p_0p_1p_2^*$ be equilateral triangles erected externally on the sides of $p_0p_1p_2$. Furthermore, we write

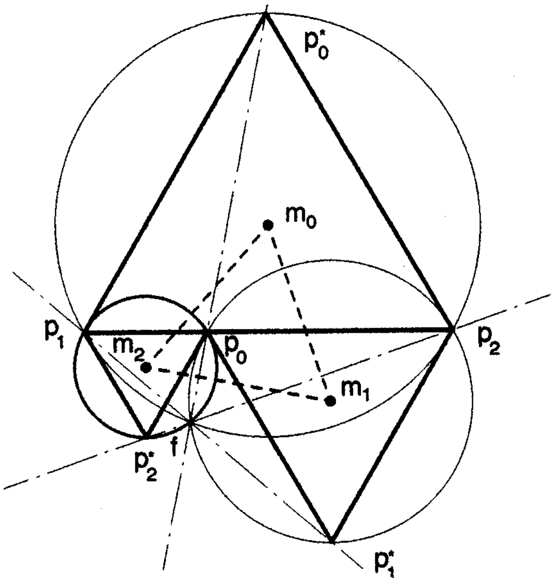

Figure 1.

m_0, m_1, m_2 for the circumcenters of these equilateral triangles. Then the following statements hold.

(1) The triangle $m_0m_1m_2$ is equilateral (Napoleon's theorem).

(2) The centroids of the triangles $p_0p_1p_2$ and $m_0m_1m_2$ coincide.

Later on we will see that these are not the only properties of the described configuration. For a large variety of further interesting properties we refer to [Ma]. For example, the circumcircles of the three erected triangles have a point f in common, and this point also belongs to the three lines connecting the points p_i and p_i^*, respectively. And the three line segments $p_i p_i^*$ have equal length. (Moreover, one might mention that a configuration with nearly analogous properties is obtained if one erects equilateral triangles *internally* on the sides of $p_0p_1p_2$, see again [Ma] and, for a generalization, Figure 5 below.)

To the best of our knowledge, until now it has not been explicitly mentioned that these relations remain true if the triangle $p_0p_1p_2$ is degenerate, i.e., if the points p_0, p_1, p_2 are collinear (Figure 1). And also the following case, where one of the inner angles of $p_0p_1p_2$ is larger than $2\pi/3$, is worth mentioning (Figure 2). E. Torricelli (cf. [To], Vol. I, Part 2, pp. 91–92) investigated this configuration in connection with a famous question of P. de Fermat (cf. [Fe], p. 153), namely to find the (unique) point minimizing the sum of distances $\|p_0 - x\| + \|p_1 - x\| + \|p_2 - x\|$ between $x \in \mathbb{E}^2$ and the given points p_0, p_1, p_2. (An extensive discussion of this problem, with many historical corrections, was given by [K-M].) However, in general the point f (mentioned above) does not coincide with

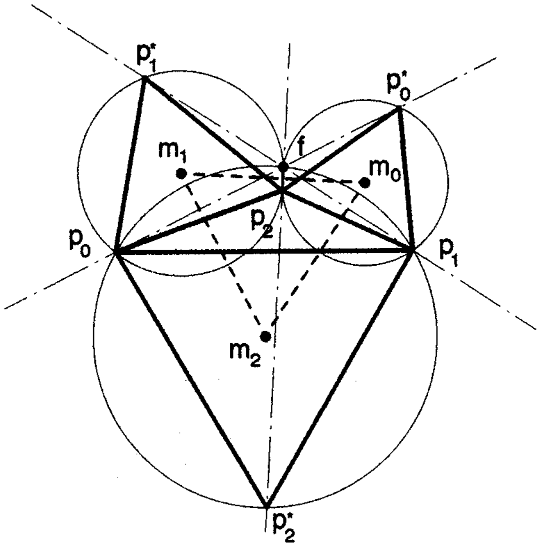

Figure 2.

this minimum point of $\{p_0, p_1, p_2\}$. This coincidence holds if and only if all inner angles of $p_0p_1p_2$ are not larger than $2\pi/3$. If one of them is larger than $2\pi/3$ (see Figure 2), then the corresponding vertex yields the minimum point. This was first noticed by B. Cavalieri, cf. [Ca], p. 508. However, here one should mention an incorrect passage in the famous book [C–R], namely in Chapter VII, Section 5.3, where R. Courant and H. Robbins give the following two remarks (which we reformulate in view of our Figure 2).

(i) The point f, from which the largest side p_1p_2 of $p_0p_1p_2$ appears under an angle of $2\pi/3$ and the smaller sides under an angle of $\pi/3$ (and which is obtained by a construction analogous to that in Figure 1), solves the following problem: to minimize the expression

$$\| p_1 - x \| + \| p_2 - x \| - \| p_0 - x \|, \quad x \in \mathbb{E}^2.$$

(ii) If all inner angles of the triangle $p_0p_1p_2$ are smaller than $2\pi/3$, then $\| p_1 - x \| + \| p_2 - x \| - \| p_0 - x \|, x \in \mathbb{E}^2$, is least at the vertex p_0.

Both these remarks from [C–R] are wrong. A counterexample to (i) is given by Figure 3: the solution proposed by [C–R] would yield a minimum value of $2 \| p_1 - f \| - \| p_0 - f \|$, say (note that the shown triangle is isosceles). By reflecting f at the line through p_1 and p_2, one obtains f^* with $\| p_1 - f^* \| + \| p_2 - f^* \| - \| p_0 - f^* \| < 2 \| p_1 - f \| - \| p_0 - f \|$, since $\| p_i - f \| = \| p_i - f^* \|$ for $i \in \{1, 2\}$; but obviously $\| p_0 - f \| < \| p_0 - f^* \|$. And a counterexample to (ii) is simply given by an equilateral triangle $p_0p_1p_2$: the solution proposed by [C–R] would yield a minimum value of $2 \| p_1 - p_0 \|$, say. But $\| p_1 - x \| +$

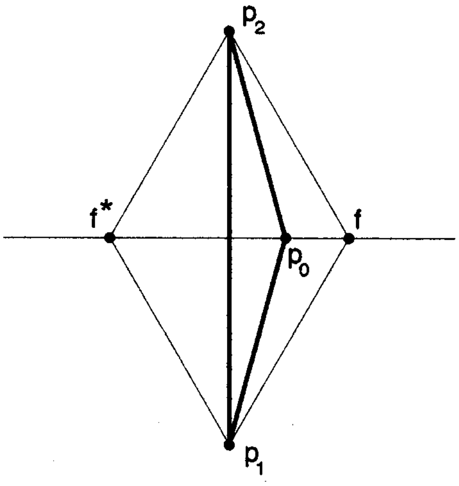

Figure 3.

$\| p_2 - x \| - \| p_0 - x \|$ is zero for $x = p_1$, as well as for $x = p_2$. The correct solution to the modified Fermat's problem (to minimize $\| p_1 - x \| + \| p_2 - x \| - \| p_0 - x \|$, $x \in \mathbb{E}^2$) was presented in [B–G], and since its description is relatively complicated, the interested reader is referred to that paper.

However, we notice that (like in the case of three inner angles $< 2\pi/3$) also in the situation of Figure 2 the point f is *isogonic* with respect to $\{p_0, p_1, p_3\}$, i.e., the lines passing through f and p_i, $i \in \{0, 1, 2\}$, pairwise enclose an angle of $\pi/3$.

For constructing isogonic points with respect to $\{p_0, p_1, p_2\}$ it is not necessary to erect equilateral triangles over $p_0 p_1$, $p_1 p_2$ and $p_2 p_0$. Having this in mind, we want to construct Torricelli's configuration in a converse manner, i.e., by starting with the given points p_0, p_1, p_2 and f. On this way we get the points m_i as centers of the circumcircles of $f p_0 p_1$, $f p_1 p_2$, $f p_2 p_0$, and the points p_i^* are obtained as intersections of these circumcircles with the lines connecting f and p_i, $i \in \{0, 1, 2\}$, respectively. As we shall see, this point of view gives an immediate motivation for suitable generalizations of Torricelli's configuration (not only with respect to the dimension, but also regarding extensions to the weighted case).

2. Torricelli's Configuration with Weights in $\mathbb{E}^n$

Let $p_0, p_1, \ldots, p_n$ be $n + 1$ points in $\mathbb{E}^n$, $n \geqslant 2$. For the sake of convenience, we assume the $(n + 1)$-tuple $\{p_0, p_1, \ldots, p_n\}$ to be affinely independent. (This could be neglected, but in the following we will ignore degenerate configurations.) Thus $\{p_0, p_1, \ldots, p_n\}$ is the vertex set of a nondegenerate n-simplex T whose facet hyperplanes we denote by H_i, where $p_i \notin H_i$ for each $i \in \{0, 1, \ldots, n\}$. In addition, we write S for the circumpshere of T.

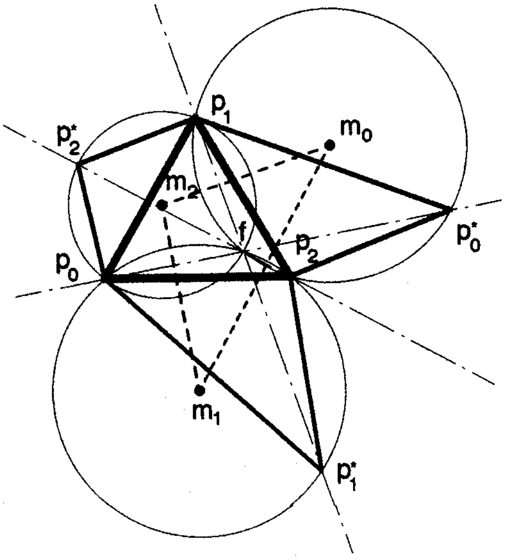

Figure 4.

Now we denote by $f \in \mathbb{E}^n$ an arbitrary point neither contained in S nor in one of the facet hyperplanes H_i. Then f and each n-tuple from $\{p_0, p_1, \ldots, p_n\}$ lie on a uniquely determined $(n-1)$-sphere, and no two of these $n+1$ spheres may coincide resp. be concentric. We write S_i for these $(n-1)$-spheres, where $p_i \notin S_i$ for each $i \in \{0, 1, \ldots, n\}$. Furthermore, m_i denotes the center of S_i, and T^* be the convex hull of $\{m_0, m_1, \ldots, m_n\}$. It is easy to see that also T^* is a nondegenerate simplex. By assumption we have $p_i \neq f$, and therefore one may consider the intersection of the line through f and p_i with S_i. If this intersection consists of two points, then that one different from f is denoted by p_i^*; and if the line passing through f and p_i is tangent to S_i, then we set $p_i^* = f$. Now one may introduce simplices (which are 'erected' over the facets of T) by

$$T_i := \text{conv}\left(\{p_0, p_1, \ldots, p_n, p_i^*\} \backslash \{p_i\}\right),$$

and the erected simplex T_i has

$$s_i := \frac{1}{n+1}\left(p_i^* + \sum_{\substack{j=0\\ j\neq i}}^{n} p_i\right)$$

as its centroid. This configuration, consisting of the points p_i, the spheres S_i having f in common, and further points and sets defined above, should be called the *n-dimensional (weighted) Torricelli configuration.* In Figures 4 and 5, one can see different possibilities for $n = 2$.

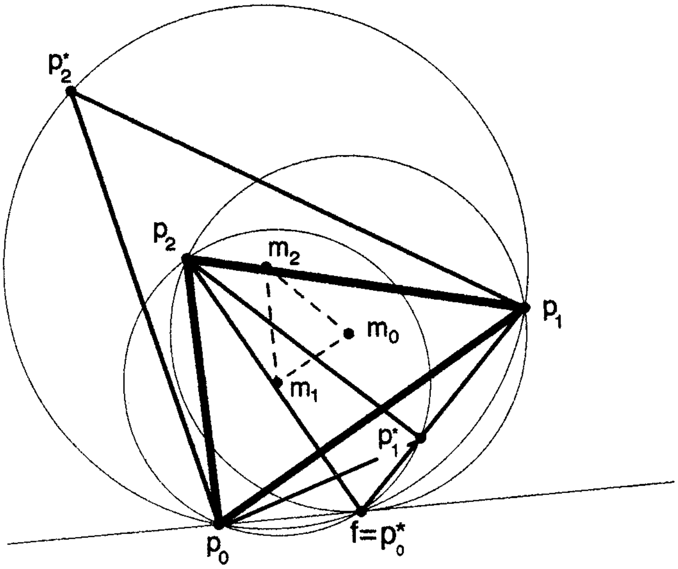

Figure 5.

Regarding the history of the planar weighted case (shown in the two figures above), the following remarks should be given. In 1877, the student E. Engelbrecht [En] proved the statement that if three directly similar triangles $p_0p_1p_2^*$, $p_1p_2p_0^*$, $p_2p_0p_1^*$ with inner angles α, β, γ are externally erected on the sides of $p_0p_1p_2$, then the lines $p_0p_0^*$, $p_1p_1^*$, $p_2p_2^*$ have a common point f with the intersection angles α, β, γ, and also the circumcircles of the erected triangles intersect at f. Although E. Torricelli investigated only the subcase of *equilateral* erected triangles (Figure 2), J. Neuberg [Neu] called also this generalized figure the *Torricelli configuration*, see [B–M], pp. 1216–1219, for further historical remarks. The 'twin point' f' of f, analogously obtained by internally erected triangles (cf. Figure 5), was investigated by A. Artzt [Ar], and already [Mü] studied the relation between f and f' by a special Cremona transform of fifth degree, cf. [Sch] for recent results about this. Furthermore, H. Uhlich [Uh] introduced angle coordinates by means of the 'weighted Torricelli figure', and the possibly most general related configuration was already studied by C.F.A. Jacobi [Ja], who only demanded that the pairs of angles of the externally erected triangles at the same vertex of $p_0p_1p_2$ are equal to each other.

Another historical line goes back to Th. Simpson, who generalized Fermat's question to *weighted distances* regarding three given points in an exercise of his book [Sim], and probably W. Launhardt [Lau] was the first person who explicitly attempted to consider the 'weighted Torricelli figure' as a resource for solving the weighted generalization of Fermat's question. More detailed discussions of this problem in view of industrial location were given by G. Pick [Pi], also with cross connections to the theory of *multifocal ellipses*, which occur as level curves regarding extensions of Fermat's problem, see [Tsch], pp. 118–121, for first investigations of these curves. In [Ya], pp. 186–189, one may find four purely geometric approaches to the weighted case of Fermat's question. Further contributions to the weighted Torricelli configuration were given or discussed by [Ri] and [Le].

3. Generalized Theorems

With the following theorems (which are higher-dimensional analogues to Napoleon's theorem and related statements) we want to clarify some relations between the point f and other objects belonging to the n-dimensional weighted Torricelli configuration. The position of f with respect to $\{p_0, p_1, \ldots, p_n\}$ is uniquely determined by the barycentric coordinates λ_i, $i = 0, 1, \ldots, n$, of f regarding the affine basis $(p_0, \ldots, p_n)$, given by

$$f = \lambda_0 p_0 + \lambda_1 p_1 + \cdots + \lambda_n p_n, \qquad \lambda_0 + \lambda_1 + \cdots + \lambda_n = 1. \tag{1}$$

Here the property $f \notin H_i$ is reflected by $\lambda_i \neq 0$, $i = 0, 1, \ldots, n$, and (1) implies

$$\lambda_0(p_0 - f) + \lambda_1(p_1 - f) + \cdots + \lambda_n(p_n - f) = o.$$

Furthermore, $f \notin \{p_0, p_1, \ldots, p_n\}$ yields

$$\sum_{i=0}^{n} \gamma_i \frac{p_i - f}{\| p_i - f \|} = o, \qquad \gamma_i := \alpha_i \| p_i - f \| \neq 0. \tag{2}$$

These real numbers γ_i (which could be determined by the numbers α_i and the scalar products $\langle p_h, p_k \rangle$) are qualified for the description of nearly all (metrical) relations in the generalized Torricelli configuration. Based on these considerations and notions we can formulate the following theorems.

THEOREM 1. *The centers m_i of the $(n-1)$-spheres S_i, $i = 0, 1, \ldots, n$, are the vertices of an n-simplex T^*. If V_i denotes the $(n-1)$-volume of the facet of T^* which is opposite to m_i, then*

$$V_h : V_k = | \gamma_h | : | \gamma_k |, \quad h, k \in \{0, 1, \ldots, n\}. \tag{3}$$

THEOREM 2. *The centroid s of the given n-simplex T is also the centroid of the mass points (s_i, γ_i^2), $i = 0, 1, \ldots, n$, i.e.,*

$$s = \left(\sum_{i=0}^{n} \gamma_i^2 \right)^{-1} \sum_{i=0}^{n} \gamma_i^2 s_i. \tag{4}$$

These theorems take a particularly simple shape if all the real numbers γ_i are equal up to their sign. This holds exactly if there exist numbers $\varepsilon_i \in \{-1, 1\}$ such that

$$\sum_{i=0}^{n} \varepsilon_i \, \frac{p_i - f}{\| \, p_i - f \, \|} = o. \tag{5}$$

Under this assumption Theorem 1 says that the n-simplex T^* has facets with equal $(n-1)$-volumes. (For $n = 2$ and $n = 3$ this even implies their congruence.) In the planar case, (5) yields the isogonic property of f with respect to $\{p_0, p_1, p_2\}$. Therefore Theorem 1 is a natural generalization of Napoleon's theorem, and Theorem 2 is a generalization of the second statement in Section 1 above.

For the sake of convenience, we set

$$\| \, p_i - f \, \|^{-1} \, (p_i - f) =: e_i, \qquad i = 0, 1, \dots, n. \tag{6}$$

Without loss of generality, the point f can be identified with the origin. Thus, setting $f = o$ we have

$$e_i = \frac{p_i}{\| \, p_i \, \|}, \qquad \sum_{i=0}^{n} \gamma_i e_i = o, \quad \gamma_i \neq 0, \quad i = 0, 1, \dots, n. \tag{7}$$

First we have to prove the following lemma:

LEMMA. *For the rank $\varrho(A)$ of the matrix $A = (\{e_0, e_1, \dots, e_n\} \backslash \{e_i\})$, $i = 0, 1, \dots, n$, we have $\varrho(A) = n$, i.e., each n-tuple from $\{e_0, e_1, \dots, e_n\}$ is linearly independent.*

Proof. It is sufficient to prove this for $i = 0$. Suppose that $\{e_1, \dots, e_n\}$ is linearly dependent. Then there exist numbers τ_i with $(\tau_1, \dots, \tau_n) \neq (0, \dots, 0)$ and

$$\sum_{i=1}^{n} \tau_i e_i = \sum_{i=1}^{n} \tau_i \, \| \, p_i \, \|^{-1} \, p_i = o.$$

Setting $\tau_i \, \| \, p_i \, \|^{-1} =: \tau_i'$ for $i = 1, \dots, n$, we get

$$\sum_{i=1}^{n} \tau_i' p_i = o, \qquad (\tau_i', \dots, \tau_n') \neq (0, \dots, 0).$$

If $\Sigma_{i=1}^{n} \tau_i' = 0$ would hold, then $\{p_1, \dots, p_n\}$ would be affinely dependent, which is impossible (since even $\{p_0, \dots, p_n\}$ is affinely independent). On the other hand, if $\Sigma_{i=1}^{n} \tau_i' \neq 0$ would hold, then by $\tau_i'' := (\Sigma_{i=1}^{n} \tau_i')^{-1} \cdot \tau_i'$, $i = 1, \dots, n$, the relations

$$\sum_{i=1}^{n} \tau_i'' p_i = o, \qquad \sum_{i=1}^{n} \tau_i'' = 1, \tag{8}$$

would follow, a contradiction to the assumption that $o = f$ does not belong to one of the facet hyperplanes of the n-simplex $T = \operatorname{conv}(\{p_0, p_1, \ldots, p_n\})$. □

Proof of Theorem 1. The $(n-1)$-sphere S_i with center m_i contains the point $f = o$ if and only if its points $x \in \mathbb{E}^n$ satisfy the equality $\| x - m_i \|^2 = \| m_i \|^2$, i.e., $\| x \|^2 - 2\langle x, m_i \rangle = 0$. Since the points p_j (with $j \neq i$) are contained in S_i, we have

$$\| p_j \|^2 - 2\langle p_j, m_i \rangle = \| p_j \|^2 - 2 \| p_j \| \langle e_j, m_i \rangle = 0$$

and by $\| p_j \| \neq 0$ for $j = 0, 1, \ldots, n$ it follows that

$$\langle e_j, m_i \rangle = \tfrac{1}{2} \| p_j \|, \quad j \neq i. \tag{9}$$

Also the scalar products $\langle e_i, m_i \rangle$ can be explicitly described. Namely, starting with

$$0 = \langle o, m_i \rangle = \left\langle \sum_{j=0}^{n} \gamma_j e_j, m_i \right\rangle = \gamma_i \langle e_i, m_i \rangle + \sum_{\substack{j=0 \\ j \neq i}}^{n} \gamma_j \langle e_j, m_i \rangle,$$

we obtain

$$\langle e_i, m_i \rangle = \tfrac{1}{2} \| p_i \| - \tfrac{1}{2\gamma_i} \delta, \qquad \delta := \sum_{j=0}^{n} \gamma_j \| p_j \| . \tag{10}$$

Following our assumptions, the quantity δ cannot vanish, since this would mean that $p_i \in S_i$. Since the rank of $(\{e_o, e_1, \ldots, e_n\} \backslash \{e_i\})$ equals n, the hyperplanes

$$H_j^* := \{x : \langle e_j, x \rangle - \tfrac{1}{2} \| p_j \| = 0\}, \quad j = 0, 1, \ldots, n, \tag{11}$$

either have exactly one point in common or present the facet hyperplanes of an n-simplex T^*. It is obvious that the latter case is true. By (9) and (10) the vertices of T^* are exactly the points m_i. Since the defining equation of the hyperplane H_i^* is given in a normalized form, one can derive the distances of the points m_i to the facet hyperlanes H_i^* (i.e., the lengths of the altitudes of T^*) by

$$h_i = \frac{1}{| \gamma_i |} \cdot \frac{| \delta |}{2}, \quad i = 0, 1, \ldots, n. \tag{12}$$

From this it follows that relation (3) in Theorem 1 is true. □

In addition one might remark that the inradius r^* of T^* (i.e., the radius of the insphere of that simplex) is determined by

$$\frac{1}{r^*} = \frac{1}{h_0} + \cdots + \frac{1}{h_n} = \frac{2}{| \delta |}(| \gamma_0 | + \cdots + | \gamma_n |). \tag{13}$$

Proof of Theorem 2. Since $f = o$, a point p_i^* from the line connecting f and p_i can be described by $p_i^* = \delta_i p_i$, and if this point belongs to S_i, we have also $\| p_i^* \|^2 - 2\langle p_i^*, m_i \rangle = 0$. With the help of $\delta_i^2 \| p_i \|^2 - 2\delta_i \langle p_i, m_i \rangle = 0$, $\| p_i \| \neq 0$, $\| p_i \|^{-1} p_i = e_i$, as well as by (10) we get the condition

$$\delta_i(\delta_i \| p_i \| - 2\langle e_i, m_i \rangle) = \delta_i \left(\delta_i \| p_i \| - \| p_i \| + \frac{1}{\gamma_i} \delta \right) = 0.$$

Following the definition of p_i^* given above, the expression between the brackets has to vanish, $\delta_i = 1 - \gamma_i^{-1} \| p_i \|^{-1} \delta$ must hold, and the equality

$$p_i^* = p_i - \frac{\delta}{\gamma_i} \cdot \frac{p_i}{\| p_i \|} = p_i - \frac{\delta}{\gamma_i} e_i \tag{14}$$

is obtained. This yields

$$s_i := \frac{1}{n+1} \left(p_i^* + \sum_{\substack{j=0 \\ j \neq i}}^{n} p_j \right) = \frac{1}{n+1} \left(\sum_{j=0}^{n} p_j - \frac{\delta}{\gamma_i} e_i \right) = s - \frac{\delta}{n+1} \cdot \frac{e_i}{\gamma_i},$$

and the relation from Theorem 2 follows by $\gamma_0 e_0 + \gamma_1 e_1 + \cdots + \gamma_n e_n = o$. □

From the results derived above one might read off further geometric properties of the generalized Torricelli configuration. For example, by (11) the facet hyperplane H_i^* of the n-simplex T^* is orthogonal to the segment $p_i f$ and intersects it in its midpoint; thus the lines through p_i and p_i^* are orthogonal to the corresponding facet hyperplanes of T^*, and by (12) and (14) the altitudes of T^* satisfy $h_i = \frac{1}{2} \| p_i^* - p_i \|$ for each $i \in \{0, 1, \dots, n\}$.

On the other hand, one might ask for geometric properties or characterizations of the simplices T_i 'erected' over the facets of T. Only for $n = 2$ these simplices have the same shape as T^*, see [Ma], Section 4, for a wide discussion of this case.

References

[Ar] Artzt, A.: Untersuchungen über ähnliche Dreiecke, die einem festen Dreieck umbeschrieben sind. *56, Jahresber. Gymnas. Recklinghausen,* Program No. 336 (1886), 1–28.

[B–M] Berkhan, G. and Meyer, W. F.: Neuere Dreiecksgeometrie, in: *Encyclop. Math. Wiss.,* Bd. III, 1. Teil, 2. Hälfte, Leipzig, 1914, pp. 1173–1276.

[B–G] Brownawell, D. and Goodman, V.: A variation of Fermat's problem, *Math. Mag.* **38** (1965), 267–276.

[Ca] Cavalieri, B.: *Exercitationes Geometricae*, Bologna, 1647.

[C–R] Courant, R. and Robbins, H.: *What is Mathematics?* Oxford Univ. Press, 1941 (German edition: *Was ist Mathematik?* Springer, 1992).

[En] Engelbrecht, E.: Planimetrischer Lehrsatz, *Arch. Math. Phys.* **60** (1877), 447–448.

[Fa] Faifofer, A.: *Elementi di Geometria* (17th edn), Venice, 1911.

[Fe] de Fermat, P.: *Œvres*, Vol. I. Ed. H. Tannery, Paris, 1891, Supplement: Paris, 1922.

[Ja] Jacobi, C. F. A.: *De triangulorum rectilineorum proprietatibus quibusdam nondum satis cognitis*. Naumburg, 1825.

[K–M] Kupitz, Y. S. and Martini, H.: Geometric aspects of the generalized Fermat–Torricelli problem, in: I. Bárány and K. Böröczky (eds), *Intuitive Geometry* (Budapest, 1995), Bolyai Soc. Math. Studies 6 (1997), 55–127.

[La] Laisant, C.-A.: Sur quelques proprietes des polygones, *Assoc. français pour l'avancement des sciences, Compte rendu* (1877), 142–154.

[Lau] Launhardt, W.: Die Bestimmung des zweckmässigsten Standortes einer gewerblichen Anlage, *Zeitschr. des Vereins deutscher Ingenieure* **26** (1882), Heft 3, 105–116.

[Le] Lester, J.: Triangles I: Shapes, *Aequationes Math.* **52** (1996), 30–54.

[Ma] Martini, H.: On the theorem of Napoleon and related topics, *Math. Semesterber.* **43** (1996), 47–64.

[Mü] Müller, H.: Über eine geometrische Verwandtschaft fünften Grades, *Math. Ann.* **2** (1870), 281–282.

[Neu] Neuberg, J.: Sur les figures semblablement variables, *Proc. London Math. Soc.* **16** (1885), 185–188.

[Pi] Pick, G.: Mathematischer Anhang zu 'A. Weber, *Über den Standort der Industrien*', Tübingen 1909 (Engl. translation by C. J. Friedrich: *Theory of the Location of Industries*, Univ. of Chicago Press, 1929).

[Ri] Rigby, J. F.: Napoleon revisited, *J. Geometry* **33** (1988), 129–146.

[Sch] Schaal, H.: Zur Verwandtschaft der Zwillingspunkte beim Dreieck, *Beitr. Algebra Geom.* **34** (1993), 177–189.

[Si] Simpson, Th.: *The Doctrine and Applications of Fluxions*, London, 1750 (improved version ed. by W. Davis, London 1805).

[To] Torricelli, E.: *Opere*, Vol. I, Faënza, 1919.

[Tsch] von Tschirnhaus, E. W.: *Medicina mentis* (Lipsiae 1695). Deutsch hrsg. von R. Zaunick, Acta Historica Leopoldina, J. A. Barth, Leipzig, 1963.

[Tu] Turner, G.: *Elementi di Geometria*, I, Palermo, 1843.

[Uh] Uhlich, H.: Altes und Neues zur Lehre von den merkwürdigen Punkten des Dreiecks, *Progr. Fürstenschule Grimma*, 1886.

[We] Weißbach, B.: Zur Übertragung des Satzes von Napoleon auf n-Simplexe, Preprint 9/1997, Fakultät für Mathematik, Universität Magdeburg, *Praxis Math.*, to appear.

[Ya] Yaglom, I. M.: *Geometric Transformations* II, Random House, New York, 1968.