PUPILS' NEEDS FOR CONVICTION AND EXPLANATION WITHIN THE CONTEXT OF GEOMETRY

Michael de Villiers, Faculty of Education, University of Durban-Westville, Private Bag X54001, Durban 4000, South Africa

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After a brief introductory analysis of the author's personal needs for conviction and explanation and their fulfillment, as well as those of mathematicians, empirical research is reported which indicates that the need for personal conviction of the majority of pupils is satisfied by quasi-empirical means, but that they nevertheless seem to exhibit an independent need for explanation not satisfied by quasi-empirical means. Exploratory results with individual children, as well as with a class, seem to indicate that it is quite feasible to utilize the latter need to introduce proof as a form of explanation, rather than as a means of verification. Drawing a parallel between this approach and the professional activity of mathematicians, it is concluded that stressing the explanatory function of proof in situations where conviction already exists, may not only make proof potentially more meaningful to pupils, but is in such cases probably more intellectually honest.

1. Introduction

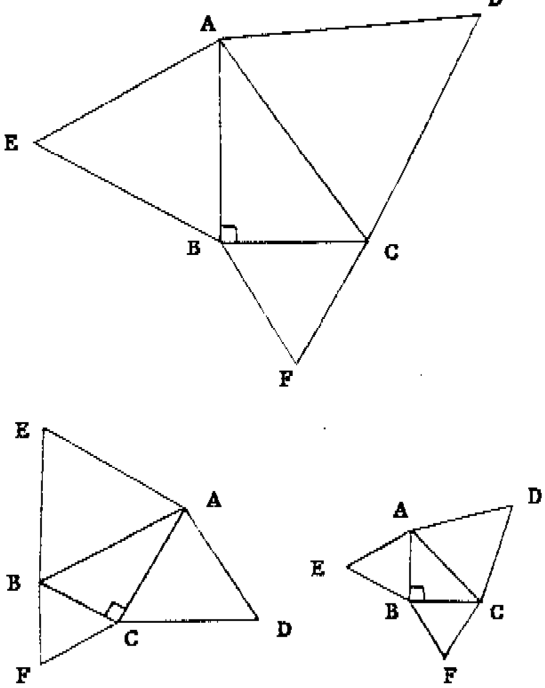

Figure 1

Consider the first figure in Figure 1 which represents a right ΔABC with equilateral triangles DAC, EAB and FBC constructed respectively on sides AC, AB and AC. Draw DB, EC and FA. What do you notice?

No doubt you noticed that EC, DB and FA are *concurrent*. What went through your mind at that point? Did you expect it or were you surprised? Do you think it will always be true? Or do you think it was just coincidence? Do you perhaps first want to test some more? If so, also draw EC, DB and FA in the bottom two figures in Figure 1. Does this confirm or refute your present suspicion?

You are now asked to be truly honest with yourself, and to ask yourself how **certain** you are that this result would be true in any right triangle. Do you largely doubt its truth or are you reasonably certain? Can you give a percentage to it? 50%, 70%, 90%, 99%, 100%? How would you make more certain? Would you be more certain if you could construct a couple more on your own?

Let's look at another example. Connect the opposite angles A with D, B with E and C with F in the four circumscribed hexagons given in Figure 2. What do you notice? Repeat the previous set of questions to yourself, and honestly reflect on the certainty or uncertainty of

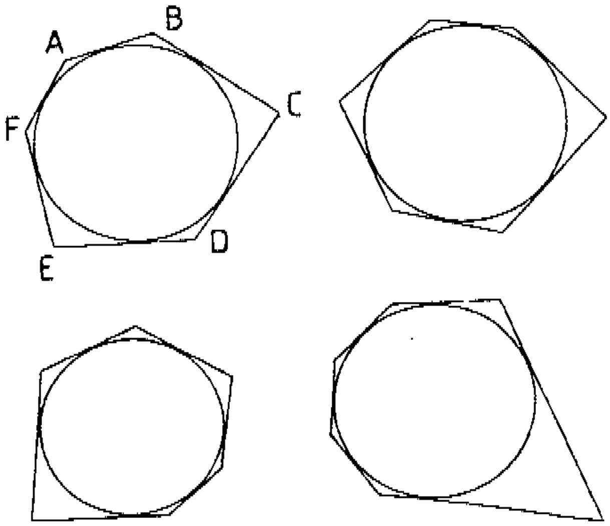

Figure 2

your personal suspicions, and your need for further conviction.

I now propose the conjecture that the majority of readers are probably now reasonably certain about the truth of both these two results. Under the further assumption of a mathematically educated and experienced audience, I am probably also correct if I assumed that none are at this stage 100% certain. However, my conjecture is that a very high level of confidence about the truth of mathematical results can sometimes be obtained without any deductive reasoning involved; i.e. by only using quasi-empirical, inductive, analogical, intuitive or heuristic means.

Let's now examine more closely the **role** or **function** of proof in situations such as the above where, in the absence of proofs, a reasonable level of certainty has already been reached.

A Personal Interlude

A strategy I have often found particularly useful when trying to analyse the **functionality** (the role or function) of certain mathematical processes and content, is to critically reflect on my own mathematical activity whilst solving problems or doing investigations of my own. For example, to introspectively analyse my own needs and preferences, and the thought processes and strategies I choose to try and fulfill them; in other words, to examine closely the rationalé or personal motivation behind all the various aspects of my mathematical activity. Although such analyses are probably highly personal and ideosyncratic, they nevertheless seem to shed light on the broad nature and meaning of mathematics in general.

Recently I was investigating the vertical line and point symmetries of functions on my own, and after considering the graphs of several cases, some of which are shown in Figure 3, I formulated the following two **dual** conjectures with their two associated corollaries:

(1) *A differentiable function $y = f(x)$ is reflective symmetric around a vertical line $x = a$ if and only if its derivative $dy/dx = f'(x)$ is point symmetric around the point* (a;0)

(2) *A differentiable function* $y = f(x)$ *is point symmetric around a point* (a; b) *if and only if its deriva-*

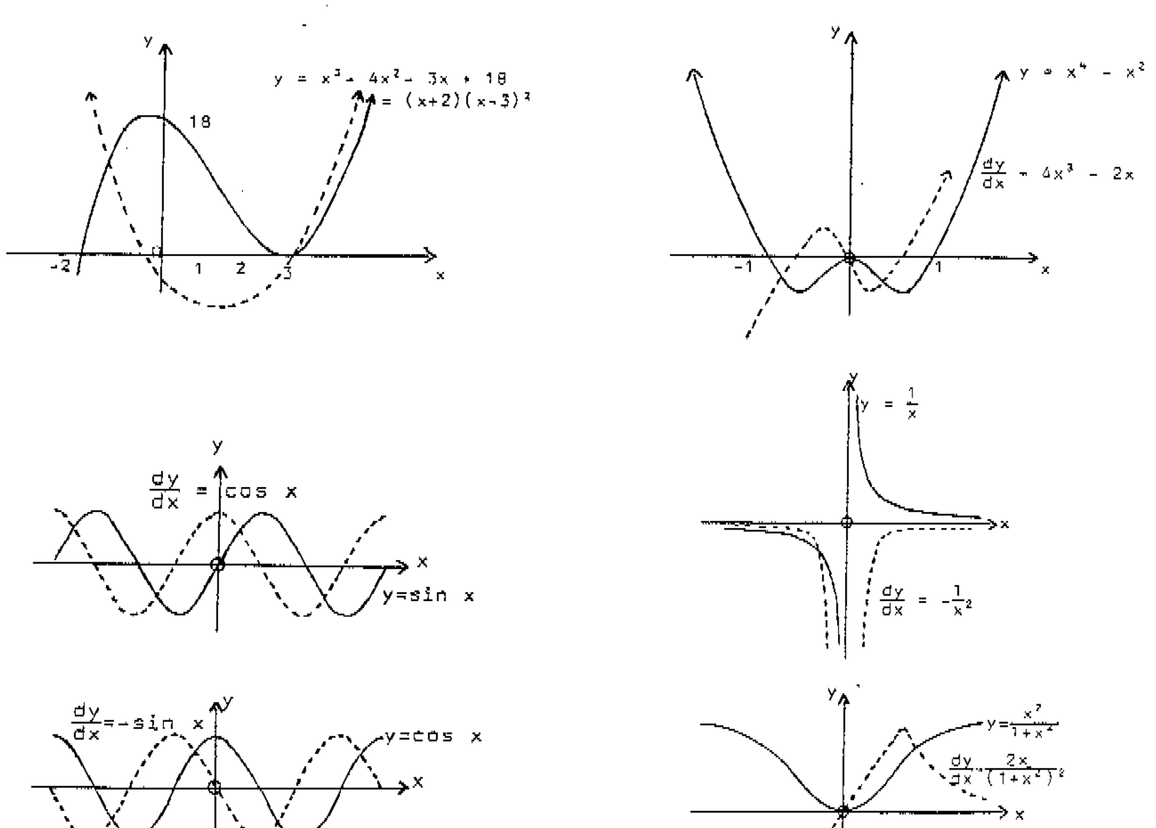

Figure 3

tive dy/dx = f′(x) is reflective symmetric around the vertical line x = a.

(3) *A differentiable function is reflective symmetric around a vertical line x = a if and only if its second derivative is reflective symmetric around the line x = a (and the first derivative is point symmetric at* (a; 0))

(4) *A differentiable function is point symmetric around a point* (a; b) *if and only if its second derivative is point symmetric around the point* (a;0).

Similar to our two introductory examples, I was fairly certain about the truth of these conjectures, simply on the basis of the wide variety of examples I had considered. But what was completely lacking was a satisfactory *explanation*. *Why* were these results true? *How* could I *explain* them? The consideration of further examples would only have increased my confidence, but it could not provide any further personally satisfactory sense of *illumination*, i.e. an *insight* or *understanding* of how and why they were the consequence of other familiar results.

Eventually I came up with intuitive geometric arguments in order to try and satisfy my personal need for explanation, a sample of which is given below.

Theorem 1

A differentiable function $y = f(x)$ is reflective symmetric around $x = a$ if and only if its derivative $dy/dx = f'(x)$ is point symmetric around the point *(a; 0)*.

Theorem 2

A differentiable function $y = f(x)$ is point symmetric around a point *(a; b)* if and only if its derivative $dy/dx = f'(x)$ is reflective symmetric around the vertical line $x = a$.

Proofs of the Forward Implications

Consider the two cases shown in Figure 4, in each case for both a continuous and a discontinuous graph. In the first case, if $y = f(x)$ is reflective symmetric around $x = a$ it means that the graph to the left of $x = a$ must fit exactly on the graph to the right of $x = a$, and vice versa. Therefore the y-values of the graphs respectively to the left and right of $x = a$ are exactly equal in sign and size, and local minima and maxima on the left correspond exactly with those on the right. However the gradient (derivative) of the graph to the left of $x = a$ is opposite in sign to the gradient (derivative) to the right of $x = a$, although equal in size. For example, the derivative of the graph on the left of $x = a$ is negative when the derivative of the corresponding part of the graph to the right of $x = a$ is positive, and vice versa. The same applies for example to the local maxima and minima of the derivative to the left and right of $x = a$, where local maxima on the left (e.g. at A) correspond exactly to local minima at the right (e.g. at A′), and vice versa. However, these are the properties of a graph which is point symmetric at *(a;0)* (see second case below), and it therefore means

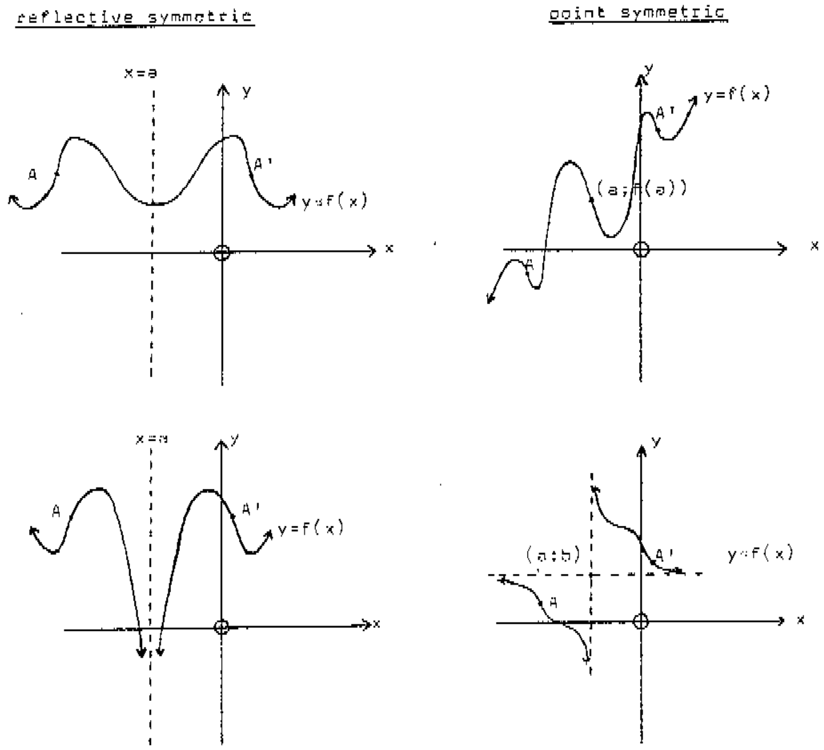

Figure 4

that the derivative in the first case is point symmetric at $(a;0)$.

Similarly, in the second case if $y = f(x)$ is point symmetric with regard to $(a;b)$, the graph to the left of $x = a$ can be made to fit exactly on the graph to right of $x = a$ by a half-turn (a rotation through 180°), and vice versa. Therefore, local minima to the left of $x = a$ correspond exactly to local maxima to the right of $x = a$, and vice versa, but the y-values of the graph to the left and right of $x = a$ are exactly equal in size and opposite in sign **only** if the point of symmetry lies on the X-axis (e.g. at $(a;0)$). (See first case above). Let's now consider the gradient (derivative) of the function $y = f(x)$ as given in the second case. Clearly in this case the gradient (derivative) of the graph to the left of $x = a$ is exactly equal to the gradient (derivative) to the right of $x = a$, not only in size but also in sign. Note that local minima and maxima of the derivative on the left correspond exactly with those on the right. For example, the local minimum on the left at A corresponds exactly with the local minimum on the right at A′. Since these are the properties of a graph which is reflective symmetric around $x = a$ (see first case above), it means that the derivative in the second case is reflective symmetry around $x = a$. Q.E.D[1].

What function did these proofs fulfill for me? Were their main function the removal of doubt or the provision of an explanation? Given my already high level of conviction, the main function of these proofs for me was clearly that of *explanation*, rather than that of [illegible] or *verification*. Granted, they did further increase my confidence, but this was totally trivial in comparison to the insight and understanding they provided. This brings me to the following:

> ***Conjecture 1**: In situations where a high level of confidence in a mathematical result is obtained prior to proof (a priori conviction), the function of the eventual proof is usually far more that of explanation than that of verification or justification.*

Some Further Evidence

Some of you may now respond by saying that my personal experience and interpretation thereof is highly ideosyncratic and not at all representative of the experience of the majority of mathematicians, nor does it correspond with the "*philosophically accepted*" view that proof's main function is that of verification/conviction. After all, I'm not first and foremost a research mathematician, but merely a mathematics educator. Although I have already in a previous article of mine (De Villiers, 1990b) proposed the above conjecture and discussed it, I will now take the liberty of briefly repeating some of those arguments for further substantiation.

That proof is not necessarily a prerequisite for conviction is borne out in the history of mathematics, i.e. by the frequent heuristic precedence of results over arguments, of theorems over proofs. For example, Gauss is reputed to have complained: "*I have had my results for a long time, but I do not yet know how I am to (deductively) arrive at them.*" Paul Halmos (1988:33) stresses the same point when he writes: "*The mathematician at work ... arranges and rearranges his ideas, and he becomes convinced of their truth long before he can write down a logical proof.*" In fact, it would appear that conviction is probably far more frequently a prerequisite for the finding of a proof, than it is the other way round. George Polya (1954:83-84) also underscores this idea as follows: "*Without such (inductive) confidence we would have scarcely found the courage to undertake the proof which did not look at all a routine job. When you have satisfied yourself that the theorem is true, you start proving it*".

To substantiate my further claim that in most cases when the results concerned are intuitively self-evident and/or are supported by convincing quasi-empirical evidence, it is not so much a question of "*making sure*" than it is a question of "*explaining why*", consider the following example. In their book **The Mathematical Experience**, Davis & Hersh (1983:363-369) discuss the "*heuristic evidence*" in support of the famous Riemann Hypothesis and then conclude that this evidence is "*so strong that it carries conviction even without rigorous proof*". However, despite this conviction, mathematicians still have an unfulfilled need for proof as a means of explanation: "*It is interesting to ask in a situation such as this, why we still feel the need for a proof ... a proof would be a way of understanding why the Riemann conjecture is true, which is something more than just knowing from convincing heuristic reasoning that it is true*". — Davis & Hersh (1983:368)

The function of a proof in the presence of "a priori" conviction, is that of explanation, not verification

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Similarly Gale (1990) recently pointed out with reference to Feigenbaum's experimental discoveries in fractal geometry, that the function of their eventual proofs by Lanford and other mathematicians was that of explanation and not really that of verification at all.

Finally, as also pointed out in De Villiers (1990b:21-22), finding out **why** a result is true, sometimes has the additional value of directly leading to a further generalization by the identification of the *essential characteristic* upon which a result depends. For example, a deductive explanation for the result shown in Figure 1 reveals the surprise that it is not necessary to have a right triangle nor equilateral triangles on the sides, but that it is true for any triangle with the triangles on the sides arranged in such a manner that the pairs of angles at each vertex are equal (e.g. consult De Villiers, 1989).

This brings me to the second part of this article, namely an exploratory investigation of pupils' cognitive needs for conviction and explanation within the context of geometry, and the ways in which the functionality of proof can be illustrated in the fulfillment of these needs.

Pupils' Cognitive Need for Conviction

Note that the following empirical results, unless otherwise stated, are from De Villiers (1990a). In order to try and find out how pupils presently gained conviction in the traditional approach to geometry education, and what their levels of conviction were, 42 geometric statements from the formally prescribed (1983) syllabus were selected and given to 519 Std.7 to Std.10 pupils (grades 9 to 12) from a Technical High School in 1986 and asked to make the following judgements with respect to each of them:

Code 1: Believe it is true from own conviction;
Code 2: Believe it is true because it appears in the textbook or because the teacher said so;
Code3: Do not know whether it is true or not;
Code 4: Do not think it is true;
Code 0 : Unanswered.

Code	0	1	2	3	4
1. The sum of the interior angles of a triangle is 180°					
Std.7	0	37.1	50.5	6.7	5.7
Std.8	0	23.1	71.4	3.4	2.1
Std.9	0.7	22.8	71.0	3.4	2.1
Std.10	0.8	18.7	74.8	3.3	2.4
TOTAL	0.4	24.8	67.9	4.0	2.9
2. The exterior angle of a triangle is equal to the sum of the opposite interior angles					
Std.7	2.9	21.8	64.8	4.8	5.7
Std.8	0.7	14.2	76.9	4.1	4.1
Std.9	2.8	11.7	78.6	4.8	2.1
Std.10	1.7	14.6	69.9	8.9	4.9
TOTAL	2.0	15.1	73.3	5.6	4.1
3. If two lines intersect, then the opposite angles are equal					
Std.7	0	29.5	57.2	4.7	8.6
Std.8	0.6	25.9	67.4	3.4	2.7
Std.9	0.7	25.5	70.3	2.8	0.7
Std.10	0.8	33.3	61.1	2.4	2.4
TOTAL	0.6	28.3	64.7	3.3	3.3
4. In an isosceles triangle, the angles opposite the equal sides, are equal					
Std.7	1.9	29.5	52.4	4.8	11.4
Std.8	0	25.9	60.5	5.4	8.2
Std.9	0.7	21.4	68.3	4.1	5.5
Std.10	0	25.2	61.0	8.9	4.9
TOTAL	0.6	25.2	61.2	5.7	7.3
5. The diagonals of a parallelogram bisect each other					
Std.7	5.7	21.0	43.8	13.3	16.2
Std.8	0.7	22.4	53.7	11.6	11.6
Std.9	0.7	18.6	64.1	6.9	9.7
Std.10	0.8	17.9	56.1	13.8	11.4
TOTAL	1.7	20.0	55.2	11.1	11.9

TABLE 1

Table 1 gives five examples of typical percentages of responses to these 42 statements (note that the given five were informally treated in Std.5 or Std.6 and then formally proved in Std.7). As Code 2 responses were in virtually all 42 cases far greater than Code 1 responses, on average two to three times greater,this leads me to the following:

Conjecture 2: The certainty or conviction of the majority of pupils with respect to prescribed statements presently seem to be based on authoritarian grounds rather than on personal conviction.

These results can probaly be attributed largely to the wide-spread dominance of the traditional approach to geometry education which is mainly a case of *"teaching by direct imposition"*, in contrast to an *"investigative approach"* in which pupils themselves explore, discover, formulate and justify geometric propositions. That the latter is not the case presently in schools, is furthermore supported by the observation that small, but significant percentages of pupils thought that the statements were false or did not know whether they were true or not.

As the traditional introductory approach to proof in geometry strongly emphasizes it as a means of verification/conviction, one should also ask how successful is this meaning transferred to pupils. For this purpose, let us briefly reflect on the following relevant data. During 1986, a total of 1959 Std.7 to Std.10 pupils from 14 different schools in two independent studies on the Van Hiele theory by Smith (1987) and De Villiers & Njisane (1987) were firstly asked whether they were certain of the truth of the following two statements (which are proved in Std.7), and secondly on what grounds they based their certainty or uncertainty (e.g. explain why you are certain or uncertain):

(1) In an isosceles triangle the angles opposite the equal sides are equal.
(2) If two parallel lines are cut by a transversal, then the alternate angles are equal.

Although 88% of all the pupils were certain of the truth of both these statements, only 7% indicated that they were certain because they could be proved, while the majority just repeated the statements as reasons or simply left the second parts unanswered. This suggests:

Conjecture 3: Only for a minority of pupils, proof seems to have the function of conviction/justification.

In order to partially substantiate my claim that the majority of teachers probably only emphasize the verification function of proof, let's consider the following question which I asked to 205 postgraduate prospective mathematics teachers at 11 universities during 1984 (De Villiers, 1987):

Why do we prove that the sum of the interior angles of a triangle is 180°, even though we can easily verify it experimentally?

In answer to this question, the majority (61%) attributed a verification function to proof (i.e. to make sure, to convince, etc.), while only a minority (7%) assigned an explanatory function to it (i.e. to explain/understand why it is true). The rest gave answers indicative of a systematization function (11%, i.e. to logically order statements), a mind development function (4%, i.e. to develop logical thinking) or uninterpretable or no responses (17%). Assuming similar viewpoints from presently practising teachers, it seems we can reasonably anticipate that the majority emphasize only the verification function of proof, and not its explanatory or other functions. In the light of this emphasis, and the failure to successfully transfer this meaning to pupils as shown in Conjecture 3, one seriously wonders how meaningful pupils are presently experiencing proof in geometry education.

In order to gain information about pupils' needs for certainty and how they themselves would choose to obtain **personal conviction**, I conducted a teaching experiment with Std.7 pupils and a series of interviews with Std.6 to Std.10 pupils during 1987 and 1988. For this purpose pupils were placed in *new* or relatively *unknown* problem situations. The following were some of the results:

- 94% of the std.7 pupils (total 32) in the teaching experiment *spontaneously* indicated that further quasi-empirical testing (i.e. construction and measurement) would satisfy their need for certainty with respect to the following geometric conjecture: "*If the midpoints of the adjacent sides of a quadrilateral ABCD are consecutively connected, then a parallelogram EFGH is formed (see Figure 5)*"

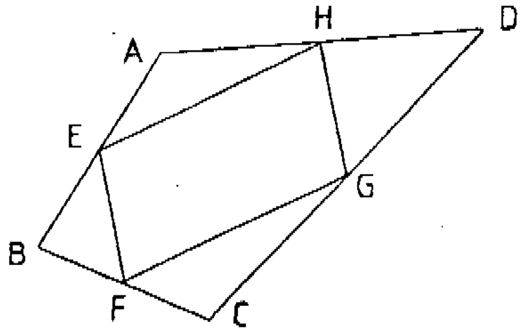

Figure 5

Typical responses were (freely translated from Afrikaans):

"*Draw a million and test*"
"*Issue a law that the whole population should draw at least two quadrilaterals to test the result*"
"*Let a class of school children each draw a variety of quadrilaterals and test the result. If there are NO exceptions, it will always be possible.*"

- 8 out of 11 Std.6 to Std.10 pupils interviewed, also spontaneously obtained certainty with respect to the above conjecture by means of construction and measurement of a number of different quadrilaterals. (5 of these pupils were from Std.8 to Std.10). Only 3 pupils therefore explicitly chose deduction as a method of verification.
- In another interview, 3 out of 7 Std.9 and Std.10 pupils obtained certainty exclusively by means of construction and measurement with respect to their own (visually formulated) conjectures that the adjacent angles and diagonals of an isosceles trapezium were equal (see Figure 6). For example consider the following extract from the interview with Lara (Std.10):

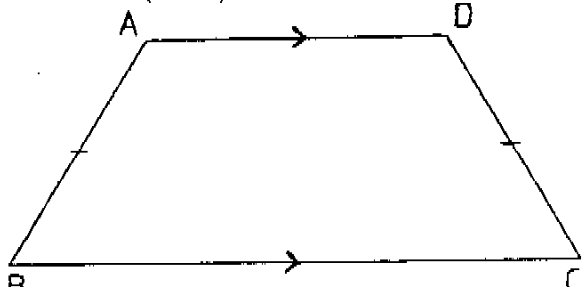

Figure 6

I: "*Is there anything in this figure that looks equal?*"
L: "*∠ A and ∠ D look the same, as well as ∠ B and ∠ C*"
I: "*Is there anything else which looks the same? ... (no answer) What about the diagonals?*"
L: "*The diagonals will be equal*"
I: "*Are you 100% certain that the angles and diagonals are equal?*"
L: "*I think I'm certain about the angles, but I'm not so sure about the diagonals ... perhaps reasonably certain about them*"
I: "*How will you make dead sure whether they are really equal?*"
L: "*By drawing them and measuring.*"

These 3 pupils also did not display an additional need for the deductive verification of these results, as illustrated by the following extract from the interview with Lynette (Std.10) who had convinced herself by construction and measurement (freely translated from Afrikaans):

I: "*If I would now give you a proof ... would that make you more certain that it is always true ... or are you at the moment sufficiently satisfied that it is always true?*"
L: (with emphasis as indicated) "*I am presently satisfied, since I observed it **myself**, and measured it myself. I am feeling satisfied because I did it **myself**.*"
I: "*You have no need that I give you a proof to convince you further?*"
L: "*No, I convinced myself.*" (sounding very firm)

Two of the other four pupils were also asked to evaluate certain alternative definitions for the isosceles trapezia. In both cases they chose quasi-empirical testing rather than deductive proof. For instance, Martin (Std.9) reacted as follows when asked how he would make sure whether any cyclic quadrilateral with equal diagonals was an isosceles trapezium or not (freely translated from Afrikaans):

M: *"One could draw a circle with two chords of equal length and intersecting, and then connecting the end points and measuring the base angles, and if the base angles are not equal, then it is not an isosceles trapezium."*

The other student was also of the opinion that a mathematician would not, like himself, make a construction to test a definition (or conjecture), but only use logical deduction (*"he would sit here with an **a** and **x** and prove it"*).

- In another interview, all 5 the Std. 7 to Std. 9 pupils chose construction and measurement to evaluate and obtain certainty about the validity of the following two conjectures *"inscribed angles in a semi-circle are 90°"* and *"angles inscribed on the same chord are equal."*

In the above examples, pupils did not always use accurate construction and measurement, but sometimes only rough drawings which they then simply evaluated visually (e.g. the first example). The above results were also not isolated observations, but has regularly been observed by the author in various other situations with different geometric conjectures. Of course, these results are not new, and is for instance confirmed by empirical research on the Van Hiele theory (e.g. Usiskin, 1982; Fuys, Geddes & Tischler, 1988). Schoenfeld (1986:243) also writes as follows: *"... most students from high school sophomore through college senior, all of whom has a full year of high school geometry, are naive empiricists ..."* From these results I now propose:

> ***Conjecture 4:** The majority of pupils spontaneously choose to satisfy their need for personal conviction in new or unknown situations by means of quasi-empirical testing.*

Pupils' Cognitive Need for Explanation and its Fulfillment

The following research was directed at exploring the *feasibility* (implementability) of an alternative approach to teaching proof. The interviews reported below should therefore be considered more as small scale *"teaching experiments"*, than clinical research in the pure sense of the word. In this regard the following results were obtained:

- Despite their certainty with respect to the conjecture that a parallelogram will always be formed by connecting the midpoints of the sides of any quadrilateral, all the Std.7 pupils in the teaching experiment exhibited an independent need for *explanation*, by responding eagerly to the question: *"You have now all convinced yourself that this conjecture is true, but would you like to know why it is true? An explanation for it?"*

The class as a whole then found the given deductive explanation in terms of the result that the line connecting the midpoints of two sides of a triangle is parallel to the third, quite satisfactory.

- The 8 Std.6 to Std.10 pupils who had chosen to use quasi-empirical testing to obtain conviction with respect to the previous conjecture in the interviews, similarly showed an additional need for explanation. For example, consider the following extract from the interview with Vicky (Std.8) (freely translated from Afrikaans):

I: *"Are you now dead sure that it will always be true?"*
V: *"Yes, now I'm certain."*
I: *"Will a logical proof make you more certain?"*
V: *"Yes, but ... not really. I'm quite certain."*
I: *"Do you rather have a need to know why it is true?"*
V: *"Yes"*
I: *"Do you really want to know why?"*
V: *"Yes, quite ... I'm inquisitive about it"* (sounding eager)

The following is another extract of an interview with two Std.6 pupils, Nadia & Lizelle:

I: *"Good, you say you are now certain. You were convinced by a number of examples. But now I would like to ask you. Do you know why it is true? ... Are you interested in knowing why it is true?"*
L: *"I would like to know"*
I: *"An explanation? A clarification?"*
N & L: *"Yes"*

The interviewer then gave a logical explanation in terms of the result that the line connecting the midpoints of two sides of a triangle, is parallel to the third side.

I: *"So does this explanation satisfy your need for explanation?"*
N & L: *"Yes"* (sounding satisfied)

- Despite displaying no further need for deductive verification, the 3 pupils who had used construction and measurement with respect to the given isosceles trapezium, still exhibited a need for explanation which had not been satisfied by their quasi-empirical approach. For example, consider the following extract from the interview with Lynette (Std.10) (freely translated from Afrikaans):

I: *"Do you perhaps have a need to know why it is true?"*
L: *"Yes ... why is it true?"* (sounding eager)
I: *"What I understand under the word 'why' is: can I explain it in terms of something else. Do you give the same meaning to it?"*
L: *"Yes"*
I: *"For which properties would you most like to have an explanation why it is true?"*
L: *"Why the diagonals are equal"*

The interviewer than gave a deductive proof of the result in terms of the congruency of triangles ABC and DCB.

I: *"Did this explanation satisfy your need?"*

L: *"Yes ... to a degree"* (sounding reasonably satisfied)

- Similarly the 5 pupils in the interview about the aforementioned circle theorems, still exhibited a need for explanation which had not been satisfied by their quasi-empirical approach; a need which was then only satisfied by the production of a logico-deductive argument. For example, consider the following extract from the interview with Lennart (Std.9) (freely translated from Afrikaans):

I: *"How would you make certain ... Or are you already dead sure that those angles are also equal?"*

L: *"No"*

I: *"Now how would you **yourself** make sure if you went home this afternoon? Suppose you had an urgent need to make sure: how would **you** make sure? What would **you** do? ... Now I want your honest opinion. I do not want what you may think I am expecting of you - I need your personal opinion."*

L: *"I would perhaps make a larger drawing and work more accurately"*

I: *"You would not perhaps try to produce a proof for it yourself?"*

L: *"Yes, if I make larger drawings, and the angles continue to come out equal, then you could prove that if you have a chord connected to two points on the circle, then those angles will be equal"*

I: *"Do you feel that if you have drawn enough accurately and measured them, and suppose you would see that they were always equal, would that be sufficient proof that it is always true? Remember I am asking you **your** opinion."*

L: *"Yes, if I have done it quite a number of times"*

I: *"Do you have a need to see a proof for it?"*

L: *"No"*

I: *"But do you perhaps wonder **why** it is true? Do you perhaps have a need to know **why** those two angles are equal, rather than just knowing that they are equal?"*

L: *"Yes, I would very much like to know **why** it is true"* (sounding eager)

I: *"Really?"*

L: *"Yes"*

The interviewer then gave a logical proof of the result in terms of the exterior angle theorem.

I: *"Now does this argument explain to you why it is true? Does it satisfy your need for explanation? ... Does it satisfy you completely?"*

L: *"Yes, it satisfies me ... I think it is sufficient"* (sounding quite satisfied)

Although Lennart (Std.9) clearly had no need for a logical proof within the context of *verification*, he nevertheless displayed an independent need for *explanation*, which was then satisfied by a logical proof. Similar observations with many other pupils, leads me to:

***Conjecture 5:** Pupils who have convinced themselves by quasi-empirical testing still exhibit a need for explanation, which seems to be satisfied by some sort of informal or formal logico-deductive argument.*

It should , however, also be pointed out that some of these pupils did indicate that the given logico-deductive argument had also further increased their confidence/certainty, although it had already been very high prior to it due to their own quasi-empirical approach. Although the supplied explanation needs to provide some insight into why the result is true, it does not seem necessary to be formal or rigourous. The author has for instance often found with younger children that informal arguments offer sufficient explanation. For example, the folding of an isosceles triangle along its line of symmetry to explain why the base angles are equal, or the rotation of a line around a fixed point causing it to swing out equal angles on both sides to explain why opposite angles in two intersecting lines are equal.

Concluding comments

What are some possible teaching implications of the aforementioned *"conjectures"*; the plausible inferences I have drawn from observing a number of special cases?

Traditionally the role and function of proof in the classroom has either been completely ignored (the fact that it is in the syllabus and will be examined is considered sufficient reason), or it has been presented as a means of obtaining certainty (i.e. within the context of verification/conviction). However, as pointed out in Conjecture 1, mathematicians often construct proofs far more for explanatory reasons than that of verification/conviction. The traditional approach of teaching proof as a means of verification in geometric situations where pupils are already convinced of the truth of the statements, therefore represents *intellectual dishonesty*, as it is not a fair reflection of actual mathematical activity and the real reasons behind it. Given that pupils become (authoritatively or quasi-empirically) acquainted with many geometric statements within the two years of informal geometry preceding Std.7 and their first introduction to proof, it is only logical to assume that most pupils would have no further need for conviction with respect to those statements previously treated.

Rather than focussing on proof as a means of verification in situations where a high level of conviction already exists, the explanatory function of proof could instead be utilized to present proof as a meaningful activity to pupils. For instance, although pupils may have no need for further (deductive) verification as shown in Conjectures 3 & 4, their need for explanation may be utilized as a meaningful context for the presentation of a proof as shown in Conjecture 5. Rather than focussing on the inadequacy of quasi-empirical methods with regard to verification in such situations and presenting proof as a means of verification, the focus could rather fall on the inadequacy of quasi-empirical methods with respect to explanation. The author has also found it useful to use the term (deductive) "*explanation*" in such situations, while the term (deductive) "*proof*" is initially restricted to those geometric situations where pupils explicitly exhibit a need for verification/conviction ("*to make sure*").

Using this alternative approach, I have found quite considerable improvement in pupils' meaningful appreciation for the role and function of proof. This was achieved by explicitly telling pupils who already have "*a priori*" conviction that in such cases we are not really interested in trying to verify the truth of the statements concerned (we already know they are true), but in trying rather to find out why they are true (i.e. upon which characteristic they depend or how they are the consequence of other results). Having myself had the past unpleasant experience in using the traditional approach where "*a priori*" conviction already existed, of seeing pupils develop an extremely negative attitude towards proof, it has been quite remarkable how this small, but very subtle change in approach, has positively influenced their attitudes. I also wish to point out that the transcribed interviews do not fully reflect all the emotional and cognitive aspects that were apparent to myself as observer (e.g. tone and changes of voice; facial expressions and other non-verbal responses). To the possibility of doubting Thomases who are not entirely convinced by the preceding evidence, I however suggest that you try it out for yourselves. What do you have to lose? Surely nothing, but think of what you could possibly gain.[2]

Stress the explanatory function of proof when conviction already exists

However, please don't get me wrong. I am not saying that we should completely neglect the verification function of proof, but that in introductory formal geometry it is probably wise to initially restrict it to statements (e.g. unknown riders) for which the pupils themselves explicitly exhibit a need for further verification/conviction. Such an approach would of course require that teachers become much more sensitive to the cognitive and emotional needs of pupils, and exploring creative new ways of fulfilling them. In other words to place themselves in their pupils' shoes, rather than the traditional approach of trying to place pupils in the teachers' shoes. The described interviews therefore also do not necessarily represent an ideal introductory approach, as pupils in my opinion should preferably construct their own explanations, rather than the teacher supplying it directly to them. Further research is also required with respect to identifying specific geometric statements that pupils find doubtful, as well as to the types and levels of arguments that satisfy pupils' respective needs for conviction and explanation.

As pupils progress in their geometry, they should therefore be confronted with a variety of appropriate situations where seemingly obvious conjectures actually turn out to be false (in this respect elementary number theory is much more appropriate than geometry, and is it regrettable that proof in our curriculum is confined to virtually only geometry). Besides the verification and explanatory function of proof, however, attention should also be given to the other functions of proof, namely *discovery, communication* and *systematization*. In other words, the alternative approach to proof I am canvassing for is not with the intention of sacrificing any fidelity in mathematics merely for pedagogical expediency, but actually the contrary: the encouragement of greater fidelity with respect to the variety of reasons behind proof. Some of the aspects of such an alternative approach are described by Chazan (1990:9) as follows:

> "*inclusion of exploration and conjecturing; presentation of demonstrative reasoning as explanatory; treatment of proving as a social activity; and emphasis on deductive proofs as part of the exploratory process, not its end point.*"

Finally, further support for the design of an investigative approach more truly reflective of mathematical practice (i.e. in which pupils actually do "*research*" like mathematicians), comes from the observation in the preceding discussions that there is not such a great difference between the needs for conviction and explanation of pupils and their fulfillment, and those of professional mathematicians. To summarise: pupils are easily convinced by authority, but so are mathematicians. For example, Goodman (1979:547) has stated: "*Certainly, if I respect a particular mathematician ... I will be willing to rely on his intuitions ... not absolutely, but to a very large extent.*" Pupils gain personal conviction by means of intuition and/or quasi-empirical testing, but so do mathematicians. Pupils empirically check already proven statements to further increase their confidence (e.g. Fischbein, 1982), but so do mathematicians when they check that there are no counter-examples possible. Finally, like mathematicians, pupils also exhibit an independent need for explanation which seems to be satisfied by the production of some sort of logico-deductive argument. Of course, there is a vast difference in the level of sophistication of a mathematician's approach and those of the aforementioned pupils, but there is nevertheless a remarkable comparison which can surely be more effectively utilized in our teaching than it has been in the past.

Footnotes

[1] Further details about these proofs, and their eventual transformation into analytical proofs in terms of the formal definition of a derivative, not for the purpose of *verification* or *explanation*, but for the purpose of *systematization*, is given in De Villiers (In press).

[2]It is greatly enheartening to note that the spirit of the new core syllabus intended for implementation from 1992/93, actually reflects some of the major aspects of this approach, namely that proof within a systematization context is postponed until Std.8, while the explanatory function of proof is to be emphasized on the Std.7 level.

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"Every mathematician knows that his best work is based not on mere reasoning but on the characteristic kind of insight he calls 'intuition'. In this sense, the word intuition refers to a faculty by which the mathematician is able to perceive properties of a structure which, at the time, he is not able to deduce. This perception can be trained, and is often quite reliable. Sometimes, when trying to work deductively, one feels like a man trying to find his way around an unfamiliar room in the dark. The mind is full of details that fail to cohere into a pattern. But then, either gradually or suddenly, one's eyes adjust to the dark, one sees dimly how the room is arranged, one knows about chairs one has not yet bumped into, and one is able to get about comfortably. It is an everyday occurrence that a mathematician knows intuitively that thus and so must be the case but does not have the vaguest idea how to go about proving it. Often, of course, he is wrong. But far more often than not he is right. Certainly, if I respect a particular mathematician and if he has had extensive experience with a particular structure, I will be willing to rely on his intuitions about the structure even in the absence of a proof — not absolutely, but to a very large extent.

Let me say at once that I am not urging the existence of an occult faculty whereby we have direct knowledge of platonic objects. Rather, I think that the mathematician's intuition is a special case of the general human ability to recognize patterns or, more specifically, to synthesize complex structures from scattered cues. Thus I think the mathematician's intuition about a particular structure is simply the result of long experience with that structure. It is not different in kind from a carpenter's 'feel' for his wood. The fact is that mathematicians are able to arrive at more or less reliable conclusions about mathematical objects without having to deduce those conclusions. Indeed, mathematical creativity is much more a matter of intuition than it is of logic."

— Nicolas Goodman (1979) in *Mathematics as an Objective Science.* **American Mathematical Monthly**, 86, p.547.