Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.11, (2022), Issue 2, pp.190-197

JHA AND SAVARAN'S GENERALISATION OF NAPOLEON'S THEOREM

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ABSTRACT. In this paper we present a novel generalisation of Napoleon's theorem to a hexagon with equilateral triangles constructed on the sides as well as a purely geometric proof of the result.

In December 2021, two Grade 11 schoolboys from India, Jayendra Jha and Sankalp Savaran, using dynamic software discovered the following apparently new generalisation of Napoleon's theorem, and posted the problem at Stack Exchange (2021).

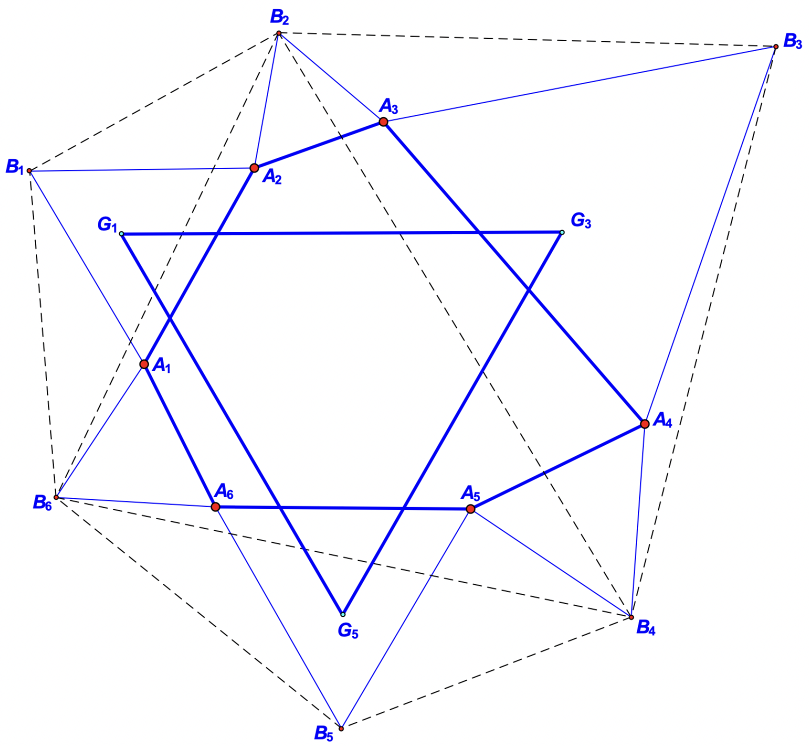

Figure 1

Key words and phrases. Napoleon's theorem, hexagon, equilateral triangles, centroids.

Jha & Savaran's Problem

Given a hexagon $A_1A_2A_3A_4A_5A_6$ with equilateral triangles constructed on the sides, either all inwardly or all outwardly, and the apexes of the equilateral triangles are labelled B_i as shown in Figure 1. If G_1, G_3 and G_5 are the respective centroids of $\Delta B_6B_1B_2$, $\Delta B_2B_3B_4$ and $\Delta B_4B_5B_6$, then $\Delta G_1G_3G_5$ is an equilateral triangle. (Similarly, the respective centroids of $\Delta B_1B_2B_3$, $\Delta B_3B_4B_5$ and $\Delta B_5B_6B_1$, form an equilateral triangle $\Delta G_2G_4G_6$.) A dynamic sketch of the above result is available for the reader to explore at:

http://dynamicmathematicslearning.com/new-napoleon-generalisation.html

Note that if, for example we let points A_1 and A_6 coincide, as well as A_2 and A_3, and A_4 and A_5, then Jha and Savaran's result reduces to Napoleon's theorem. Jha and Savaran's problem can be easily proved with the following two useful theorems:

Theorem 1 (120° rhombus theorem)

Given a quadrilateral $ABCD$ with four equilateral triangles ΔABE, ΔBCF, ΔCDG, and ΔDAH constructed on its sides, all inwardly or all outwardly. The outward construction is shown in Figure 2. If T_1 and T_3 are the respective centroids of ΔABE and ΔCDG, and G_2 and G_4 are the respective centroids of ΔEGF and ΔEGH respectively, then $T_1G_2T_3G_4$ is a rhombus with angles 120° and 60° (equivalently: $\Delta G_2T_3G_4$ and $\Delta T_1G_2G_4$ are two adjacent equilateral triangles).

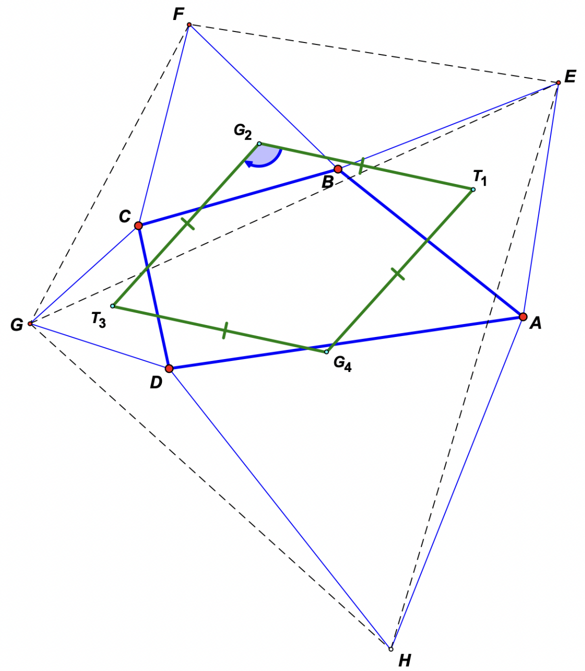

Figure 2

Remarks:

- Theorem 1 is very useful, important in its own right, and quite challenging to prove. We think it deserves to be better known, hence we'd like to propose the mentioned catchy name for it.
- For proving Theorem 1 it suffices to show $T_1G_2 = G_2T_3$ and $\angle T_1G_2T_3 = 120°$.

Pompe's Theorem
Given a hexagon $ABCDEF$ with $AB = BC$, $CD = DE$ and $EF = FA$, and angles $\angle B + \angle D + \angle F = 360°$, then the respective angles of ΔBDF are $\angle B/2$, $\angle D/2$ and $\angle F/2$.
A proof of Pompe's theorem is given in Pompe (2016) as well as in De Villiers & Humenberger (2021), and a proof of Theorem 1 will be given further on. We shall now use the two aforementioned theorems to prove Jha and Savaran's generalisation of Napoleon's theorem.

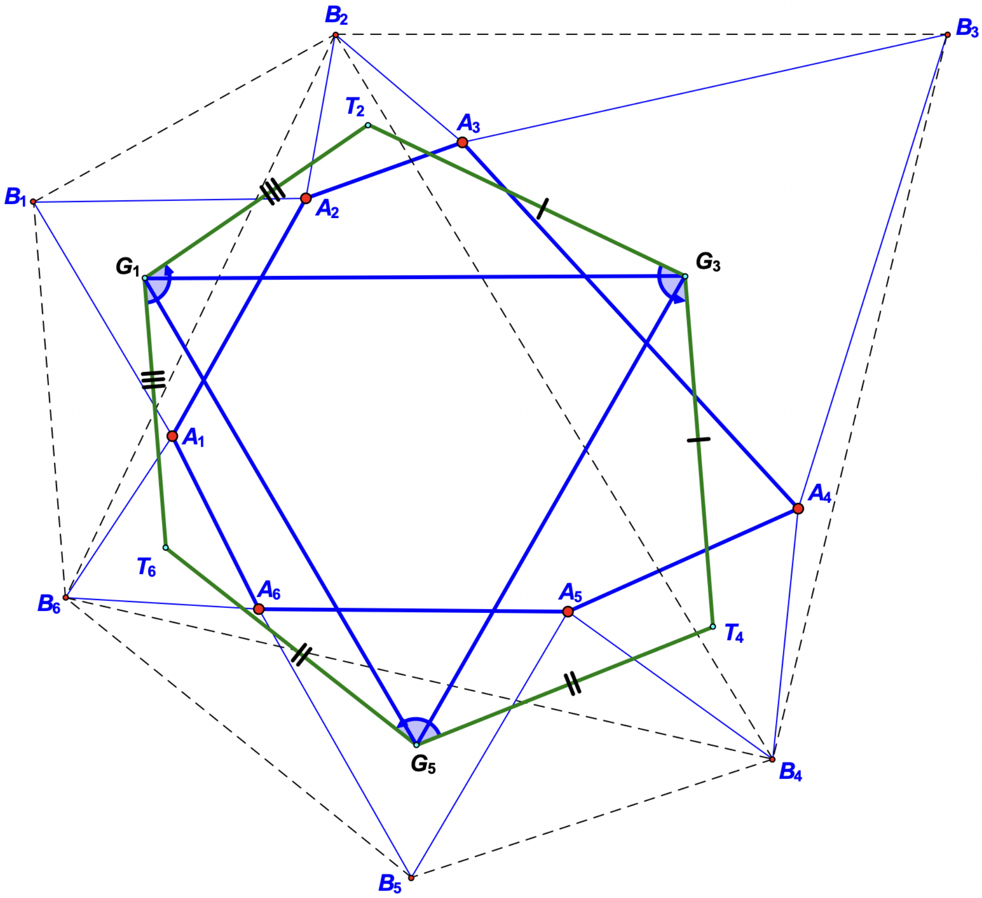

Figure 3

Proof of Jha & Savaran's result
Consider Figure 3, where T_2, T_4 and T_6 are the respective centroids of the equilateral triangles on sides A_2A_3, A_4A_5 and A_6A_1. Then according to Theorem 1, $\angle T_2G_3T_4 = \angle T_4G_5T_6 = \angle T_6G_1T_2 = 120°$, and $T_2G_3 = G_3T_4$, $T_4G_5 = G_5T_6$, and $T_6G_1 = G_1T_2$. The hexagon $G_1T_2G_3T_4G_5T_6$ therefore meets the requirements of Pompe's theorem, and hence, it follows that $\angle G_1G_3G_5 = \angle G_3G_5G_1 = \angle G_5G_1G_3 = 60°$, and this completes the proof. Alternatively, after proving that $\angle T_2G_3T_4 = \angle T_4G_5T_6 = \angle T_6G_1T_2 = 120°$, and $T_2G_3 = G_3T_4$, $T_4G_5 = G_5T_6$, and $T_6G_1 = G_1T_2$ it's also easy to see that one can construct equilateral triangles on the sides T_2T_4, T_4T_6, and T_6T_2 so that their centroids coincide with G_1, G_3 & G_5. Hence, directly from Napoleon's theorem, it then follows that $\Delta G_1G_3G_5$ is an equilateral triangle.

Remarks:

- Note that the result also holds if the equilateral triangles lie inwardly, and even if the original hexagon A_i is concave or crossed. This can easily be checked by dragging a dynamic convex configuration into these cases, and is left for the reader to verify, as well as to prove (by possible modification of the above proofs).
- Also note with reference to Figure 1, that the six centroids G_1, G_2, G_3, G_4, G_5 & G_6, respectively of $\Delta B_6B_1B_2$, $\Delta B_1B_2B_3$, $\Delta B_2B_3B_4$, $\Delta B_3B_4B_5$, $\Delta B_4B_5B_6$ & $\Delta B_5B_6B_1$, form a hexagon with opposite sides equal and parallel (De Villiers, 2007; Lord, 2008). Hence, from the half-turn symmetry of the formed parallelo-hexagon, the two equilateral triangles $\Delta G_1G_3G_5$ and $\Delta G_2G_4G_6$ are congruent via a half-turn around the point of symmetry of the parallelo-hexagon. (In addition, its point of symmetry is also the common centroid of the six centroids G_i).

Proof of Theorem 1
We shall now make use of the following two Lemmas to prove Theorem 1.

Lemma 1 (Four equilateral triangles)
Let ΔABC be a triangle and G its centroid. Now construct two concentric circles c_1 and c_2 with centre G, one through A and the other through B, and two equilateral triangles $\Delta AA'A''$ and $\Delta BB'B''$ with vertices on the respective circles (here A' denotes the point rotated around G by 120° counter-clockwise, analogously with B'). Then the triangles $\Delta CA''B'$ and $\Delta CB''A'$ are equilateral too (Figure 4).

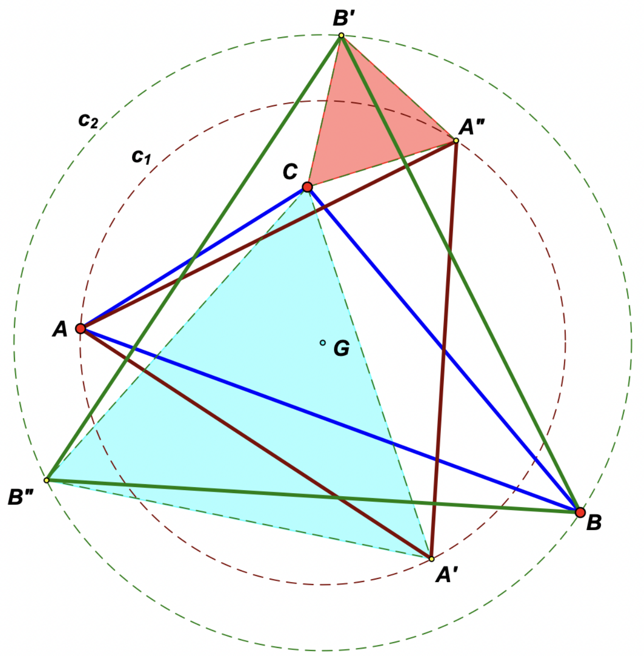

Figure 4

Proof: The first important thing to notice here is that the lines $A'A''$ and BC bisect each other. To see the reason for this phenomenon, extend AG beyond G by the half of its length, then it is clear that the new endpoint is on the one hand the midpoint of $A'A''$ and on the other hand, the midpoint of BC. Therefore, $A'BA''C$ is a parallelogram, and analogously $B''CB'A$. Hence, $B'A = CB''$ and $BA'' = A'C$. But we also know by 120° rotations with centre G that $B'A = B''A' = BA''$ and this means that $\Delta CB''A'$ is equilateral. And analogously one can prove that $\Delta CA''B'$ is equilateral.

Lemma 2
Let ΔABC be a triangle (blue) with centroid S. Then at point C an arbitrary equilateral triangle is suspended (red), the other vertices are D, E. With the line segments AD and BE two further equilateral triangles are constructed (green), F and G are their centroids. And then with the line segment FG a fourth equilateral triangle is constructed (black; the orientation of the equilateral triangles is as shown in Figure 5). Finally, one can observe: The centroids of this last (black) equilateral triangle and the initial ΔABC coincide (S).

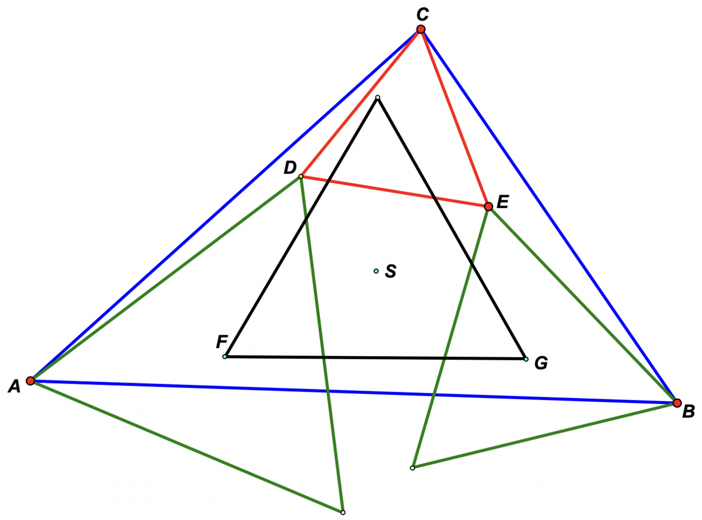

Figure 5

Proof: We have to show that the map $F \mapsto G$ is a rotation with centre S and angle 120° (counter-clockwise).
This map consists of three components:

(1) $F \mapsto D$ is a spiral similarity with centre A, angle 30° (counter-clockwise) and factor $\sqrt{3}$,
(2) $D \mapsto E$ is a rotation with centre C and angle 60° (counter-clockwise),
(3) $E \mapsto G$ is a spiral similarity with centre B, angle 30° (counter-clockwise) and factor $\frac{1}{\sqrt{3}}$.

In the composition of these components the product of the factors equals 1, the sum of the rotation angles is 120° (counter-clockwise). Hence, it is a counter-clockwise rotation by 120°, we have to determine its centre. The centre is supposed to be S, thus we have to prove: S is the fixed point of this map (Figure 6).

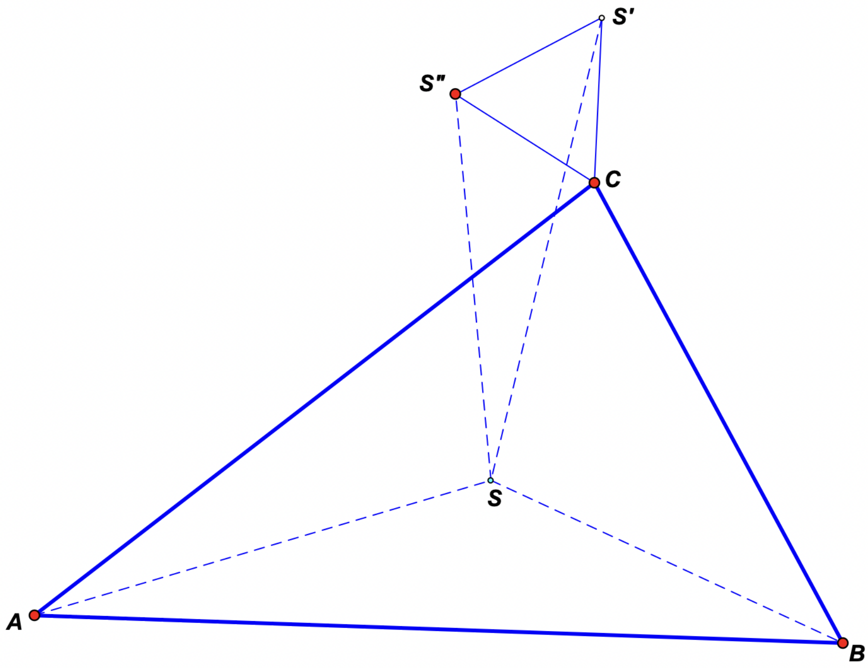

Figure 6

The spiral similarity (1) maps S onto S', where we have $\angle S'SA = 120°$. Let S'' be the inverse image of S under the spiral similarity (3), where we have $\angle S''SB = 120°$. According to Lemma 1 we know that $\Delta CS'S''$ is equilateral, hence $\angle S'CS'' = 60°$, and S' maps under the rotation (2) onto S'', altogether this means: Under the composition of the three mentioned maps we have $S \mapsto S$, thus S is the fixed point and the centre of the counter-clockwise rotation by $120°$ which yields $F \mapsto G$.

Proof of Theorem 1

The points T_1, T_3, G_2 in Theorem 1 correspond to the points G, F, S in Lemma 2, the points G, E, F of Theorem 1 correspond to the points A, B, C in Lemma 2, and the points C, S', S'' of the proof of Lemma 2 correspond to the points C, A'', B' of Lemma 1.

Concluding Remarks

Note that the hexagon $G_1T_2G_3T_4G_5T_6$ in Figure 3 is also a Haag hexagon, which is defined as a hexagon $ABCDEF$ with $AB = BC$, $CD = DE$, $EF = FA$ and $\angle B = \angle D = \angle F = 120°$ (Schattschneider, 1990, p. 90). Not only does a Haag hexagon tile, but also as the famous Dutch artist Escher apparently discovered (Rigby, 1991), it has its main diagonals concurrent (in this case, G_1T_4, T_2G_5, and G_3T_6). This concurrency follows easily from Jacobi's generalisation of the Fermat-Torricelli point of a triangle (De Villiers, 2014). Another interesting result related to the configurations in Figures 1 and 3 is the following theorem (Oai, 2015): If equilateral triangles are constructed all outwardly or all inwardly, on each side of a hexagon, then the midpoints of the segments connecting the centroids of opposite triangles form another equilateral triangle.

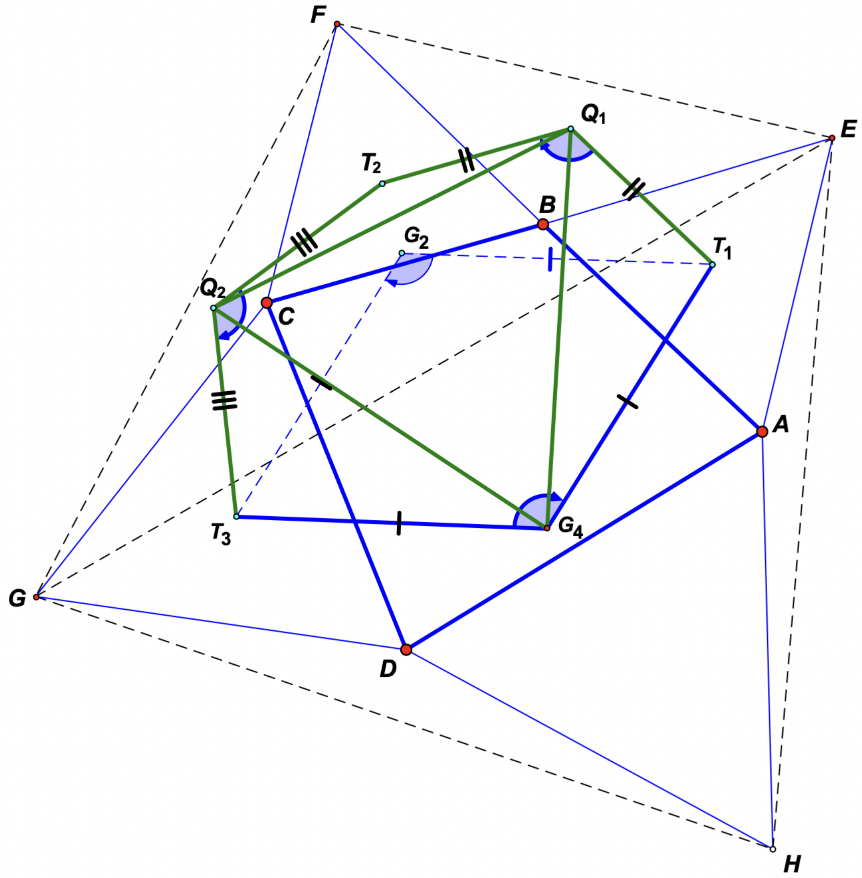

Figure 7

Several other equilateral triangles are also embedded in the configuration of Theorem 1 shown in Figure 2. For example, consider Figure 7 where T_2 is the centroid of ΔBCF and Q_1 and Q_2 are the respective centroids of ΔEBF and ΔFCG. From a known result related to Napoleon's theorem (attributed to Dao Thanh Oai by Bogomolny, date unknown), we then have $\angle T_1Q_1T_2 = \angle T_2Q_2T_3 = 120°$, $T_1Q_1 = Q_2T_2$ and $T_2Q_2 = Q_2T_3$. Since $\angle T_3G_4T_1 = 120°$ and $T_3G_4 = G_4T_1$ from Theorem 1, it follows that $\Delta Q_1Q_2G_4$ is also equilateral.

Lastly, since all equilateral triangles are directly similar, many more (in fact, infinitely many) different equilateral triangles can be associated with the configurations in Figures 1 and 2 by the use of a fundamental theorem of similarity, namely: If F and F' are any two directly similar figures with points P in F corresponding to points P' in F', and the lines PP' are divided in the ratio of $r : (1-r)$, that is, at points $P'' = (1-r)P + rP'$, then the new figure F'' formed by the points P'' will be directly similar to F and F' (DeTemple Harold, 1996; Fried, 2021). For example, if the midpoints ($r = \frac{1}{2}$) of the line segments connecting corresponding points of a pair of equilateral triangles on opposite or adjacent sides of a quadrilateral or hexagon are constructed, another equilateral triangle is formed.

It is now left to the reader to identify other equilateral triangles embedded in the configurations of Figures 1 and 2.

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