Maths bite: averaging polygons

In [1], David Wells mentions the following pretty result:

'Take any hexagon, and find the centres of gravity of each set of three consecutive vertices. These immediately form a hexagon whose opposite sides are equal and parallel in pairs.'

What he does not explicitly mention is that this is the second in a chain of similar results, the first being the familiar observation that the midpoints of the sides of any quadrilateral form the vertices of a parallelogram. Retaining Wells' description, the general result is:

'Take any 2n-sided polygon, and find the centres of gravity of each set of n consecutive vertices. These form a 2n-sided polygon whose opposite sides are equal and parallel in pairs.'

The proof is a 'one-liner'. Let $a_1, a_2, \ldots, a_{2n}$ be the position vectors of consecutive vertices $(A_k)$ of the 2n-sided polygon and continue the labelling cyclically so that $a_{2n+i} = a_i$ for $1 \leq i \leq n - 1$. The position vectors of the vertices $(B_k)$ of the derived 2n-sided polygon are then given by $b_k = \frac{1}{n}(a_k + a_{k+1} + \ldots + a_{k+n-1})$ for $1 \leq k \leq 2n$. It follows that $b_{k+1} - b_k = \frac{1}{n}(a_{k+n} - a_k)$ and, replacing $k$ by $n + k$, that $b_{n+k+1} - b_{n+k} = \frac{1}{n}(a_{2n+k} - a_{n+k}) = \frac{1}{n}(a_k - a_{n+k})$. Thus the opposite sides $B_kB_{k+1}$ and $B_{n+k}B_{n+k+1}$ are equal and parallel; indeed, they are parallel to the diagonal $A_kA_{k+n}$ of the original 2n-sided polygon.

Finally, it is worth noting that $(A_k)$, $(B_k)$ and the vertices of each parallelogram $B_kB_{k+1}B_{n+k}B_{n+k+1}$ all share a common centre of gravity since $\sum a_k = \sum b_k$ and $\frac{1}{4}(b_k + b_{k+1} + b_{n+k} + b_{n+k+1}) = \frac{1}{2n} \sum a_k$.

Reference


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