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ORDER OF OPERATIONS: THE MATH AND THE MYTH

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In a recent paper Bay-Williams & Martinie (2015) claimed that the order of operations is not an arbitrary convention, and that it is a myth that the mathematical community arbitrarily decided on the order of operations. They then go on to try and substantiate their argument as follows: "Let's look at an example that will help use see *why* multiplication precedes addition: $4 + 3 \times 5$
Because multiplication is repeated addition, we can rewrite this expression with an equivalent expression: $4 + 5 + 5 + 5$ …"

Though I've seen similar presentations in many South African textbooks and materials in the primary school, this is mathematically not correct at all! The authors are clearly unaware that they are already *assuming* here that the number sentence $4 + 3 \times 5$ follows PEMDAS/PEDMAS[1]; and that the 3 x 5 in the number sentence *must be done first*, and hence implies $5 + 5 + 5$. This is a totally CIRCULAR line of argument! One might as well say that $4 + 3 \times 5$ MEANS that one has to calculate from left to right, first doing the addition and then the multiplication. In conventional notation, in other words, that it means $(4 + 3) \times 5$, which means that the number sentence represents adding up $(4 + 3)$ five times, i.e. $(4 + 3) + (4 + 3) + (4 + 3) + (4 + 3) + (4 + 3)$. The authors Bay-Williams & Martinie (2015) don't seem to realize that they are essentially just saying 'it is so' because 'it is so'!

I think we are so used to the convention about what $3 + 4 \times 7$ means that often people think it just HAS to be like that. It's kind of ingrained into us, and most people think it just has to be so, and don't question it at all.

[1] Called BODMAS in South Africa and UK (where BIDMAS is also sometimes used).

The authors' use on p. 23 of the following example to supposedly demonstrate to students why multiplication should precede addition is unfortunately also a *subterfuge*:
"She wrote 8 + 3 × 5 + 7 on the board and said, "The Haktaks have one stack of eight coins, three stacks of five coins, and one stack of seven coins. Tell me how many coins the Haktaks have."

Again PEMDAS/PEDMAS is already assumed here for the number sentence, and Ms. G just conveniently **chose** or **selected** a real world context that fit her *preconceived interpretation* of the meaning of the number sentence. Again the illustration is entirely circular, and does NOT in the least explain why multiplication is done before addition. In fact, it is a subterfuge and entirely fake, misleading students completely from the real reason why there has to be a defined order of operations, namely, that of avoiding *ambiguity* (in number sentences, as well as in symbolic algebra).

For example, try out the following three examples with your students[2] as an experiment:
"Write 8 + 3 × 5 + 7 on the board and say, "The Haktaks have five stacks of eight plus three coins in each stack, and one stack of seven coins. Tell me how many coins the Haktaks have."
"Write 8 + 3 × 5 + 7 on the board and say, "The Haktaks have five plus seven stacks of eight plus three coins in each stack[3]. Tell me how many coins the Haktaks have."
"Write 8 + 3 × 5 + 7 on the board and say, "The Haktaks have a stack with eight coins and three stacks of five plus seven coins in each stack. Tell me how many coins the Haktaks have."

In the first example, children are likely to *understand* the context and will calculate a solution as 5 times 11 plus 7 to obtain 62, while in the second example, they will get 12 times 11 to obtain 132. In the third example, they will get 12 times 3 and adding 8, will obtain an answer of 44. In other words, it is the **context** that *determines* the calculation

[2] Assuming here, of course, students who have NOT yet learnt or been taught the standard order of operations.
[3] This is of course, a very clumsy way of really saying twelve stacks of eleven coins in each.

procedure, and NOT the representation of the calculation procedure that determines the order of operations.

If instead the children had initially been asked to represent the calculation procedure used in each context, rather than having the number sentence $8 + 3 \times 5 + 7$ **imposed** on them *a priori*, they might **represent** each case in several different ways. From personal experience of the implementation of the problem-centered approach of the University of Stellenbosch in the 1980's in the elementary school, where students were allowed to develop their own algorithms to solve 'real world' problems as well as use their own written representations, it is also hardly likely that young children themselves would choose to model or represent their calculations by the same number sentence that the teacher chose to artificially impose upon them. The following is an example of what they themselves might write to represent the order of operations in each case[4]:

$3 \times 5 \longrightarrow 15 + 8 \longrightarrow 23 + 7 \longrightarrow 30$

$8 + 3 \longrightarrow 11 \times 5 \longrightarrow 55 + 7 \longrightarrow 62$

$5 + 7 \longrightarrow 12 \quad 8 + 3 \longrightarrow 11 \quad 12 \times 11 \longrightarrow 132$

$5 + 7 \longrightarrow 12 \times 3 \longrightarrow 36 + 8 \longrightarrow 44$

Let us now discuss the real reason behind the need to have a well-defined order of operations. Let us take the first example of the authors, namely: $4 + 3 \times 5$.

What does this MEAN? In the *absence* of any predetermined rule or definition this number sentence can be interpreted in two completely different ways, namely:

1) $4 + 3 \longrightarrow 7 \times 5 \longrightarrow 35$
2) $3 \times 5 \longrightarrow 15 + 4 \longrightarrow 19$

In fact, as is probably known to most readers, there are some calculators that calculate from left to right exactly in the order in which the operations have been written down as

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[4] In this approach, it was suggested to children to use flow diagram 'arrows' to illustrate their calculation, and not the 'equal' sign – see for example, Murray. Olivier & Human (1998).

in 1), and are called *sequential logic* calculators. However, with an *algebraic logic*[5] calculator, the given answer would be 19, as such a calculator follows the assumed convention of order of operations, and will do the multiplication before the addition[6].

It might be an interesting exercise to readers, as I've done on occasion, to ask people on the street or a non-mathematician in the staffroom, what the answer is to $4 + 3 \times 5$, and one is likely to find a large proportion of people who will naturally use sequential logic. The problem is that the number sentence $4 + 3 \times 5$ is *ambiguous*, and in mathematics we try to avoid confusing situations like this where one can interpret something in two entirely different ways to obtain different 'answers'. This is not allowed in mathematics, and to avoid such ambiguity, we need to define a unique, unambiguous way that avoids this confusion. This can be achieved by defining a specific order of operations such as PEMDAS/PEDMAS that will inform us how to interpret such a number sentence. One must also realize here that our whole algebra of the real number system is based on the same *assumed notation* (convention) that the algebraic expression $a + b \times c$ follows PEMDAS/PEDMAS, and that multiplication has priority over addition[7].

I realize it is often very difficult for teachers to stand back and appreciate this fundamental point since the order of operations has become so much part of them, they take it is a 'given' rather than as a mere convention. To better perhaps better understand and appreciate that it is a convention is to step outside of school arithmetic and high school algebra. For example, consider any two binary operators, say # and *, defined in some way over a closed set of numbers, critically think about what the following means? $a \# b * c$

[5] Calculators that are called 'scientific' usually use algebraic logic.

[6] Some calculators use Reverse Polish Notation (RPN) like the HP calculators where the operators come after the operands, for example to determine 4 + 3, one would key in the following sequence: 4; 3; +, and starting 'inside' brackets and working out, one does need not to key in any brackets, or use the = sign.

[7] Though somewhat whimsical, it is quite conceivable that on another planet somewhere in the universe, should there be other intelligent life, that let alone them having a different notation for arithmetic and algebra, they may have chosen a different 'order of operations' to ours.

Does it mean, first carry out the binary operation #, and then the *? Or does it mean, do it the other way round? Clearly there is ambiguity here until we agree on what this notation MEANS! I find that doing something like this often helps teachers to realize that we need to AGREE on what $a \# b * c$ means; otherwise we'll have chaos[8]!

One could also use set theoretic, logical or Boolean algebra operators to explain it as I've often done with teachers who struggle with this. For a set theoretic example:
what does $A \cup B \cap C$ mean?

Suppose set A = {1}, set B = {3; 4} and set C = {4}. If we use SEQUENTIAL logic, processing from left to right then we get $A \cup B$ = {1; 3; 4} and its intersection with C then gives {4}. However, if we first do the intersection of B with C we get {4} when joined ($\cup$) with A gives {1; 4}. So again we get an AMBIGUITY!!! So it is necessary to decide how to deal with something ambiguous like $A \cup B \cap C$ in a CONSISTENT way. If we choose that we will interpret such a sequence in a sequential way then we obviously need to DIFFERENTIATE the other case by using a suitable notation; e.g. say curly brackets as in $A \cup \{B \cap C\}$, which will then mean that the brackets $\{\}$ have 'priority' over $\cup$, and that we then first do the intersection before we do the union. So it's all about avoiding ambiguity, and of trying to achieve precise, accurate communication!

It may interest some of the readers of *AMESA News* that several years ago I developed possible worksheets for elementary school students on the order of operations around the ambiguity issue (De Villiers, 1992), and have tried them out with children on a couple of occasions with some success. In these sheets, children are led to the confusing situation, where the same number sentence produces two different answers, which then obviously needs some form of resolution. Though I'm not suggesting that exactly the same approach be used, a similar kind of 'cognitive conflict' creating approach could go some

[8] Obviously, it gets even more confusing if more binary operators, say @ and & are added into the mix to obtain longer and longer strings, and one has no clear set of rules about the order in which these operators should be used.

way towards developing a sound mathematical and deeper conceptual understanding of why the order of operations is a necessary convention.

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