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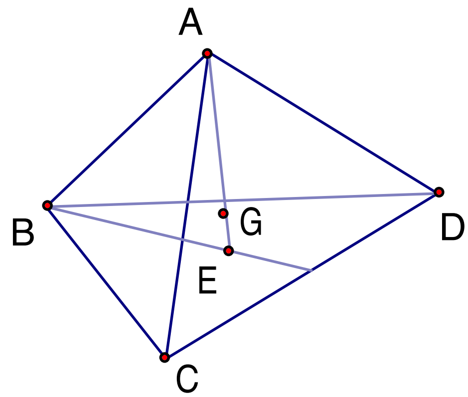
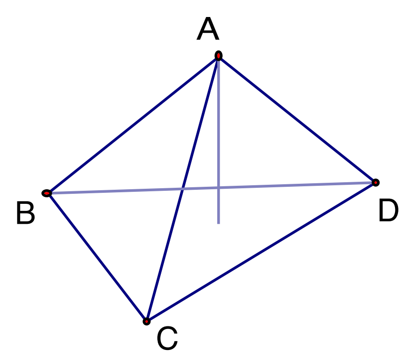
Definition 3: A median of a tetrahedron is the join of a vertex to the centroid of the opposite face.

Theorem 3: The four medians of a tetrahedron meet at the centre of mass G.

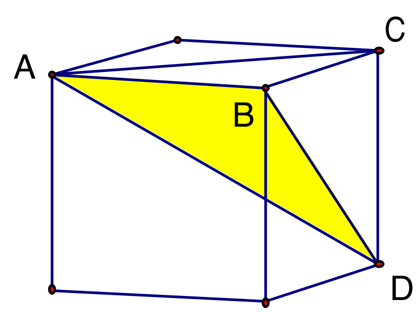


Proof: Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be coordinate vectors of A, B, C, D. Then $\mathbf{e} = \frac{1}{3}(\mathbf{b} + \mathbf{c} + \mathbf{d})$ is the centroid E of BCD. Let G be the point $\mathbf{g} = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$. Then G lies on the median AE because $\mathbf{g} = \frac{1}{4}\mathbf{a} + \frac{3}{4}\mathbf{e}$. Similarly G lies on all four medians.

To verify that G is the centre of mass of the tetrahedron, note that the line containing BE divides triangle BCD into two triangles of equal area. Therefore the plane containing ABE divides the tetrahedron into two subtetrahedron of equal volume (they also have the same height). Therefore the centre of mass lies in this plane, and similarly in the plane containing ACE, and hence on AE. Similarly the centre of mass lies on all the medians, and hence is G.



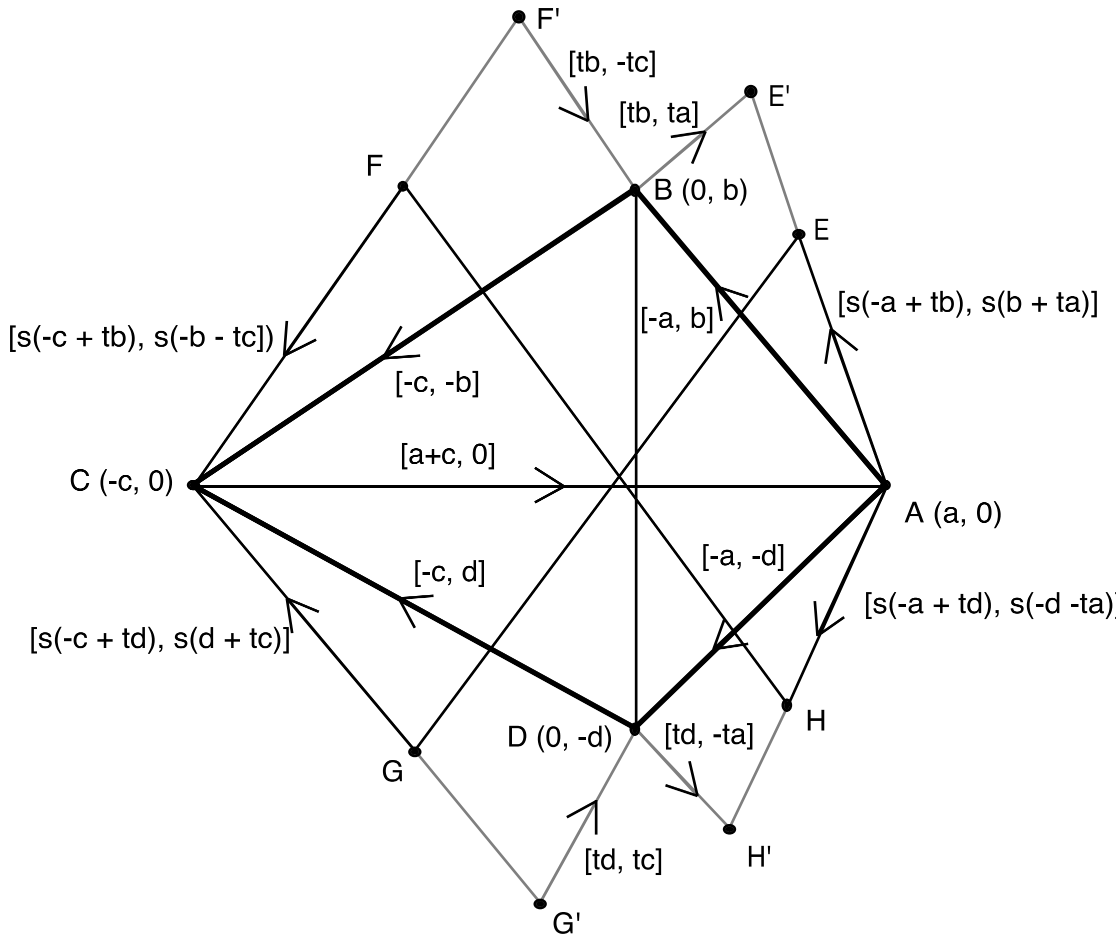
Definition 4: The altitude of a tetrahedron through A is the line perpendicular to BCD.



In general the four altitudes of a tetrahedron do not meet. It suffices to give a counter-example. Consider Dehn's tetrahedron $ABCD$ inscribed in a cube as shown. The altitudes through A, D are AB, CD which do not meet.

4. This result was experimentally discovered with *Sketchpad* by Michael de Villiers in 2004, though it is not known whether it is original. Two different, elegant proofs by Michael Fox from Leamington Spa, Warwickshire, UK, e-mail: mdfox@foxleam.freeserve.co.uk ; are given below.

(a) Proof by vectors



Given: Quadrilateral *ABCD* has perpendicular diagonals and triangles *AEB*, *AHD*, *CFB*, *CGD* are similar.

To prove: The lengths *FH*, *EG* are equal.

Proof: We use vector displacements, taking A as $(a, 0)$, B as $(0, b)$, C as $(-c, 0)$, D as $(0, -d)$.

Triangle $BE'A$ is right angled at B, with E' lying on AE.

Displacement **AB** is $[-a, b]$, so, if $t = \tan(\angle BAE)$, then **BE'** = $[ta, tb]$, and **AE'** = **AB** + **BE'** = $[-a + tb, b + ta]$.

If $\frac{AE}{AE} = s$, then **AE'** = s **AE** = $[s(-a + tb), s(b + ta)]$.

The other vector displacements follow similarly.

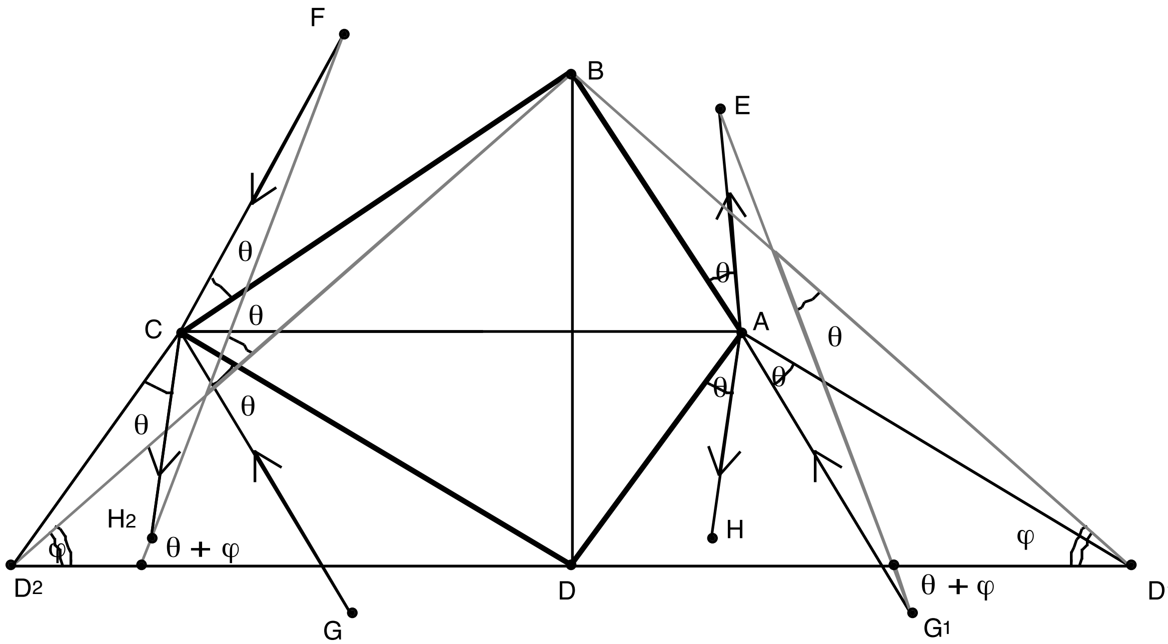
Then **FH** = **FC** + **CA** + **AH** = $[a + c, 0] + s\,[-a - c + t(b + d), -b - d - t(a + c)]$;

and **GE** = **GC** + **CA** + **AE** = $[a + c, 0] + s\,[-a - c + t(b + d), b + d + t(a + c)]$.

Since the x-components in each expression are equal, and the y-components are equal and opposite, the displacements have equal magnitudes, i.e. the lengths *FH*, *GE* are equal.

We also see that these lines are equally inclined to the diagonals *AC*, *BD*.

(b) *Proof vectors with transformations*



Proof: Since **FH** = **FC** + **CA** + **AH**, and **GE** = **GC** + **CA** + **AE**, both containing **CA**, the result would follow if the vectors **FC** + **AH**, **GC** + **AE** had equal magnitudes and were equally inclined to **CA**.

Translate DC, GD by vector **CA**. Their images are D_1A, G_1A, thus **GC** + **AE** = $\mathbf{G_1A}$ + **AE** = $\mathbf{G_1E}$.

In the similar triangles, let $\angle EAB = \ldots = \theta$, and $\frac{AE}{AB} = k$, then the spiral similarity (k, θ) takes AB to AE and CD to CG. Thus AD_1 goes to AG_1. It follows easily that this similarity takes $\mathbf{D_1B}$, i.e. $\mathbf{D_1A}$ + **AB**, to $\mathbf{G_1A}$ + **AE**, that is, $\mathbf{G_1E}$.

If we translate DA, HA by vector **AC** we obtain D_2C, H_2C, and a similar argument shows that the spiral similarity $(k, -\theta)$ takes $\mathbf{D_2B}$ to $\mathbf{H_2F}$.

Now triangle BD_2D_1 is isosceles: BD is $\perp$ to D_2D_1, and D is the midpoint of that line, so BD_2, BD_1 are equal in length and are equally inclined to D_2D_1, i.e. to CA, although with opposite angles of rotation, say φ.

Consequently, the opposite spiral similarities give images G_1E, H_2F that are equal in length, and equally inclined to CA, at an angle $\theta + \varphi$, and the result follows.

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"*Statistics can be used to support anything - especially statisticians*" - Franklin Jones, **Women's Realm**