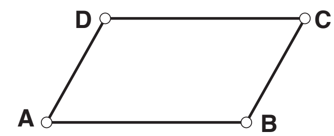
Name(s):

Parallelogram Angle Bisectors

In this activity, you will investigate the kind of quadrilateral formed by the angle bisectors of a parallelogram.



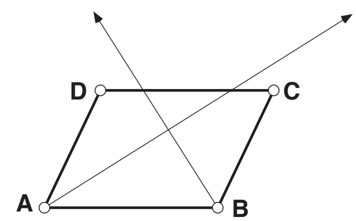
CONJECTURE

Open the sketch **Parallelogram.gsp.**

1. Drag different vertices of your quadrilateral *ABCD*. What features of quadrilateral *ABCD* make you sure it is a parallelogram? Take measurements if you wish.

Press the buttons that show each of the four angle bisectors of parallelogram *ABCD*.

Press the button that shows the quadrilateral formed by the intersection of the angle bisectors.



2. Drag different vertices of quadrilateral *ABCD*. What kind of quadrilateral do you think *EFGH* is? (Measure its angles if necessary.)

3. Try to drag vertices of *ABCD* until all sides of *EFGH* are equal. What do you find?

4. What happens to *EFGH* when *ABCD* is a square?

5. What happens to *EFGH* when *ABCD* is a rhombus?

CHALLENGE Provide proofs of your conjectures from Questions 2–5 above.

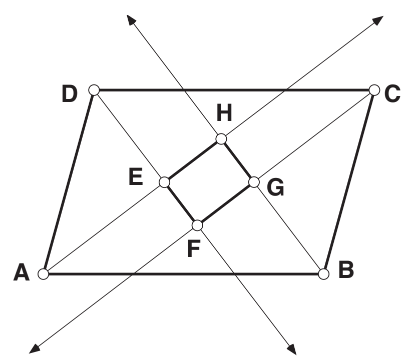
PROVING

In the preceding section, you constructed the angle bisectors of a parallelogram, then formed quadrilateral *EFGH* at the intersections of the bisectors. You should have found that

- *EFGH* is always a rectangle. (Sometimes *EFGH* is a square, which is a special case of a rectangle, and sometimes *EFGH* is a point, which you can think of as a rectangle with sides of length 0.)
- *EFGH* is a square only when *ABCD* is a rectangle. However, when *ABCD* is a square or a rhombus, the angle bisectors meet in one point.

The hints that follow will help you prove these observations.

PROVING *EFGH* IS A RECTANGLE



6. Let $m\angle DAB = 2x$ and $m\angle ABC = 2y$. Express $m\angle AHG$ in terms of x and y.

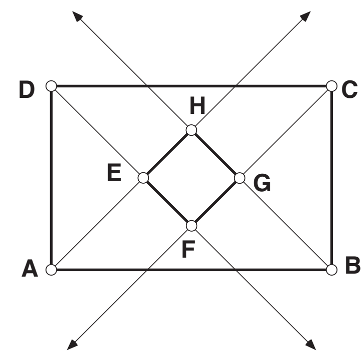
7. What can you say about the sum of the measures of angles *DAB* and *ABC*? Why?

8. Write your observation from Question 7 as an expression in terms of x and y and simplify.

9. How is Question 8 related to Question 6? What does this tell you about $m\angle AHG$?

10. Explain whether the same argument applies to the other angles of *EFGH*.

PROVING *EFGH* IS A SQUARE WHEN *ABCD* IS A RECTANGLE



11. When *ABCD* is a rectangle, what can you say about $\overline{FD}$ and $\overline{FC}$? Why?

12. What can you say about triangles *DAE* and *CBG*? Why?

13. What does this imply regarding $\overline{ED}$ and $\overline{GC}$?

14. From Questions 11–13, what can you therefore say about $\overline{FE}$ and $\overline{FG}$? Why?

15. What does this tell you about *EFGH*?

Further Exploration

1. Explain why rectangle *EFGH* is a point only when *ABCD* is a rhombus.

2. In a new sketch, construct the angle bisectors of any quadrilateral and investigate the type of quadrilateral they form. Prove your observations.

PARALLELOGRAM ANGLE BISECTORS

(PAGE 101)

This worksheet follows up on some of the earlier worksheets by presenting proof as a means of explanation, verification, and discovery. Students are given slightly less direction in constructing proofs so that they can gradually become more independent.

Prerequisites: Knowledge of congruency, properties of parallel lines and rectangles. For the optional Further Exploration section at the end, knowledge of the properties of cyclic quadrilaterals is also required (e.g., from the Cyclic Quadrilateral and Cyclic Quadrilateral Converse activities in this book).

Sketch: Parallelogram.gsp. An additional sketch is **Quad Bisectors.gsp.**

CONJECTURE

1. Opposite sides are parallel.
2. *EFGH* is a rectangle.
3. *EFGH* becomes a square (has all sides equal) when *ABCD* is a rectangle.
4. *EFGH* becomes a point.
5. *EFGH* becomes a point.

CHALLENGE This provides students with an opportunity to attempt their own proofs.

PROVING *EFGH* IS A RECTANGLE

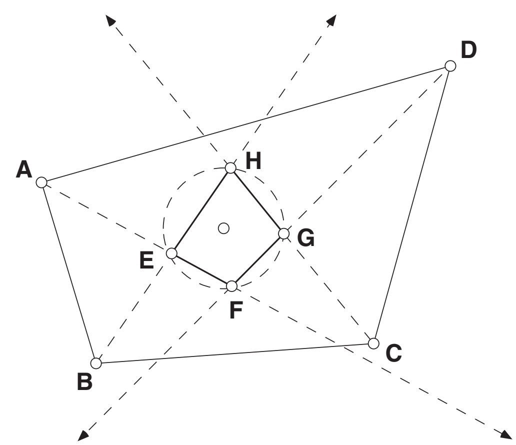
6. $m\angle AHG = 180° - x - y$ (sum of the measures of the angles of triangle *AHB*).
7. They are supplementary, since $\overline{AD} \parallel \overline{BC}$.
8. $2x + 2y = 180°$, which simplifies to $x + y = 90°$.
9. $m\angle AHG = 180° - (x + y) = 180° - 90° = 90°$.
10. Yes, the other angles are also 90°, and therefore *EFGH* is a rectangle.

PROVING *EFGH* IS A SQUARE WHEN *ABCD* IS A RECTANGLE

11. $FD = FC$, since $m\angle FDC = 45° = m\angle FCD$.
12. Triangles *DAE* and *CBG* are congruent (SAA).
13. Therefore, $ED = GC$.
14. $FD - ED = FC - GC$; therefore, $FE = FG$.
15. A rectangle with one pair of adjacent sides equal is a square.

Further Exploration

1. The angle bisectors of a rhombus coincide with its diagonals, which meet in only one point.
2. The additional sketch **Quad Bisectors.gsp** could be used for this exploration. The angle bisectors of any quadrilateral form a cyclic quadrilateral (provided it is not a circum quadrilateral, that is, circumscribed around a circle, since its angle bisectors are obviously concurrent). Students should be able to work from the preceding argument to prove this observation (for the convex case) as follows.



Proof (Convex)

$$m\angle BHC = 180° - \left(\frac{1}{2}\angle B + \angle C\right) \text{ and}$$

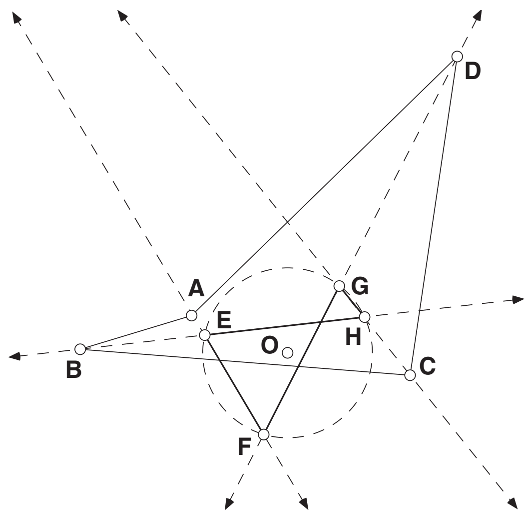
$$m\angle AFD = 180° - \left(\frac{1}{2}\angle A + \frac{1}{2}\angle D\right)$$

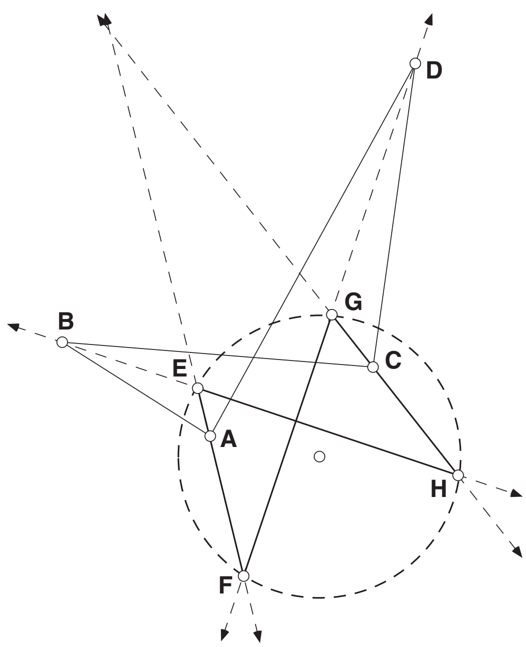
Therefore,

$$\angle BHC + \angle AFD = 360° - \frac{1}{2}\left(\angle A + \angle B + \angle C + \angle D\right)$$

$$= 360° - 180° = 180°$$

Therefore, *EFGH* is a cyclic quad.





In a case in which *ABCD* is concave or crossed, *EFGH* becomes a crossed quadrilateral, so the proofs need to be adapted using directed angles and require knowledge of the properties of crossed quadrilaterals (for a proof, see de Villiers 1996, 191–192).

PARALLELOGRAM SQUARES (PAGE 104)

This activity reinforces the idea that constructing a logical explanation (proof) can be perceived as an intellectual challenge after a result is found to be true experimentally. This activity can also be done later if you feel that it may be too challenging for students at this stage.

Prerequisites: Side-angle-side condition for congruent triangles; symmetry properties of parallelograms, rhombuses, and squares, as well as their hierarchical relationships.

Sketch: Para Squares.gsp. Additional sketches are **Aubel 1.gsp** and **Aubel 2.gsp.**

CONJECTURE

1. They are squares.
2. It is a parallelogram.
3. *EFGH* is a square.
4. Yes, it remains a square.
5. Yes, it remains a square. Note that the squares now lie on the "inward" sides of the parallelogram.

INVESTIGATING FURTHER

6. The whole configuration maps onto itself under a half-turn, and therefore *EFGH* must also be a parallelogram. (A parallelogram is the only quadrilateral with half-turn symmetry.)
7. Triangles *HAE* and *HDG* are congruent.
8. 90°.
9. $m\angle EHG = 90°$, since $\overline{GH}$ is rotated onto $\overline{EH}$.

PROVING

10. $m\angle HAE = 90° + m\angle BAD$, since $m\angle HAD$ and $m\angle EAB$ both equal 45°.
11. $m\angle BAD + m\angle ADC = 180°$, since they are co-interior angles between the two parallels $\overline{AB}$ and $\overline{DC}$.
12. $m\angle HDG = 360° - (45° + 45° + m\angle ADC)$

    $= 360° - (90° + 180° + m\angle BAD)$

    $= 90° + m\angle BAD.$