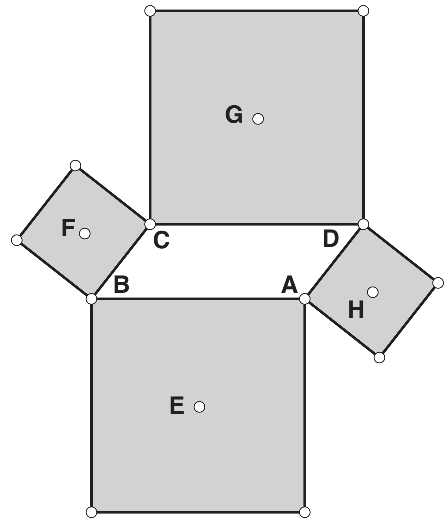
Name(s):

Parallelogram Squares

In this activity, you will investigate the kind of quadrilateral formed by connecting points *E, F, G,* and *H* in the construction shown here. The construction contains special quadrilaterals.

CONJECTURE



- Open the sketch **Para Squares.gsp.** Drag points in your sketch to familiarize yourself with this construction.

1. Describe the four shaded quadrilaterals.

2. Describe quadrilateral *ABCD.*

- Use the **Polygon** tool to construct quadrilateral *EFGH.*

3. Drag any of the points *A*, *B*, *C*, and *D*.
What kind of quadrilateral is *EFGH*? Measure some angles and sides to check your conjecture.

4. Drag *A* so that $\overline{AD}$ is parallel to $\overline{AB}$. Does your conjecture from Question 3 still hold?

5. Drag *A* across $\overline{CD}$ so that the shaded quadrilaterals overlap. Does your conjecture from Question 3 still hold?

CHALLENGE Provide a proof of your conjecture from Question 3.

Investigating Further

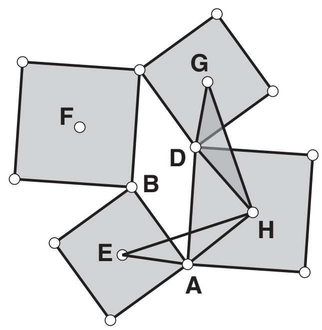
You have observed that quadrilateral *EFGH* is always a square, but you may not yet be able to explain *why* this is true. This section will help you investigate the problem further to come up with some ideas for a proof.

- Press the *Half-turn* button.

6. What do you notice about the original construction? Describe its symmetry. Since a quadrilateral *EFGH* has the same symmetry, what can you already conclude about it?

- Press the button that shows triangles *HAE* and *HDG*.

7. What do you notice about these two triangles? Drag points and take measurements to explore experimentally. Then try to explain your observations logically.



Carefully select the interior and choose **Rotate** from the Transform menu. Type the number of degrees you wish to rotate and click OK.

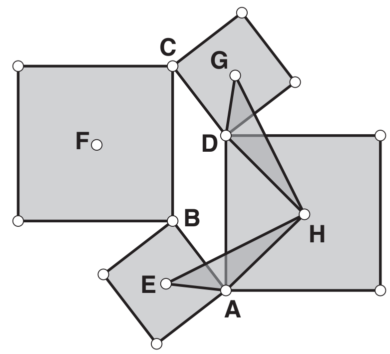
- Double-click on point *H* to mark it as a center of rotation. Then rotate the interior of ΔHDG so that it lies inside ΔHAE.

8. How many degrees did you rotate around *H* to map ΔHDG onto ΔHAE?

9. What can you now conclude regarding $\angle EHG$, and consequently about *EFGH*?

CHALLENGE Try to use your observations from Questions 6–9 to construct a proof that quadrilateral *EFGH* is a square. Discuss your thoughts with a partner. If you get stuck, read the hints that follow.

PROVING



The development of a logical argument to defend a mathematical result is often perceived as an intellectual challenge by mathematicians. This is your chance to rise to that challenge!

Follow the steps below to construct a proof of your original conjecture.

10. Explain the relationship between $\angle HAE$ and $\angle BAD$. (*Hint:* Drag point *B* until $\overline{AB}$ is parallel to $\overline{AD}$.)

11. Explain the relationship between $\angle BAD$ and $\angle ADC$.

12. Describe $\angle HDG$ in terms of $\angle BAD$. (*Hint:* Look at the angles surrounding point *D*.)

13. What can you conclude from Questions 11 and 12?

14. What can you say about the corresponding sides *EA* and *GD* of triangles *HAE* and *HDG*? Why?

15. What can you say about the corresponding sides *AH* and *DH*? Why?

16. From Questions 13–15, what can you conclude about triangles *HAE* and *HDG*, and therefore about the corresponding sides *HE* and *HG*?

17. What can you conclude about quadrilateral *EFGH* at this point?

18. What can you say about $\angle AHD$? Why?

19. What can you therefore say about $\angle EHG$? Why?

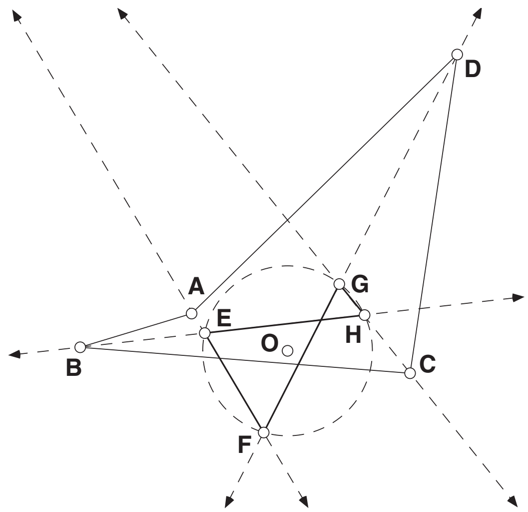
20. What can you conclude about quadrilateral *EFGH* now? Why?

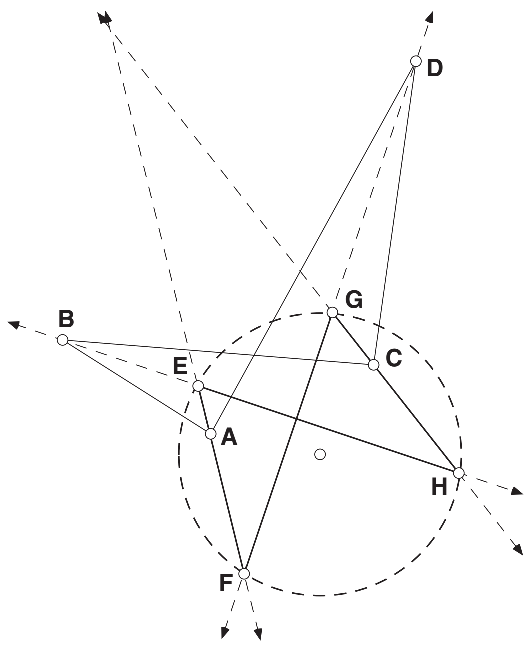
Present Your Proof

Look over Questions 6 and 10–20. Now write a proof of your original conjecture in your own words. You may include a demonstration sketch to support and explain your proof.

Further Exploration

1. In Question 5, you saw that if the squares lie inwardly and overlap (rather than lying outwardly), the result still holds. Can you adapt your proof for this configuration?
2. What type of quadrilateral is formed by the centers of squares constructed on the sides of an isosceles trapezoid? Can you explain your observation?
3. What type of quadrilateral is formed by the centers of the squares constructed on the sides of a kite?





In a case in which *ABCD* is concave or crossed, *EFGH* becomes a crossed quadrilateral, so the proofs need to be adapted using directed angles and require knowledge of the properties of crossed quadrilaterals (for a proof, see de Villiers 1996, 191–192).

PARALLELOGRAM SQUARES (PAGE 104)

This activity reinforces the idea that constructing a logical explanation (proof) can be perceived as an intellectual challenge after a result is found to be true experimentally. This activity can also be done later if you feel that it may be too challenging for students at this stage.

Prerequisites: Side-angle-side condition for congruent triangles; symmetry properties of parallelograms, rhombuses, and squares, as well as their hierarchical relationships.

Sketch: Para Squares.gsp. Additional sketches are **Aubel 1.gsp** and **Aubel 2.gsp.**

CONJECTURE

1. They are squares.
2. It is a parallelogram.
3. *EFGH* is a square.
4. Yes, it remains a square.
5. Yes, it remains a square. Note that the squares now lie on the "inward" sides of the parallelogram.

INVESTIGATING FURTHER

6. The whole configuration maps onto itself under a half-turn, and therefore *EFGH* must also be a parallelogram. (A parallelogram is the only quadrilateral with half-turn symmetry.)
7. Triangles *HAE* and *HDG* are congruent.
8. 90°.
9. $m\angle EHG = 90°$, since $\overline{GH}$ is rotated onto $\overline{EH}$.

PROVING

10. $m\angle HAE = 90° + m\angle BAD$, since $m\angle HAD$ and $m\angle EAB$ both equal 45°.
11. $m\angle BAD + m\angle ADC = 180°$, since they are co-interior angles between the two parallels $\overline{AB}$ and $\overline{DC}$.
12. $m\angle HDG = 360° - (45° + 45° + m\angle ADC)$

$$= 360° - (90° + 180° + m\angle BAD)$$
$$= 90° + m\angle BAD.$$

13. Therefore, $m\angle HAE = m\angle HDG$.
14. $EA = GD$, since squares E and G are congruent (on opposite sides of parallelogram).
15. $AH = DH$ (property of a square).
16. Triangles HAE and HDG are congruent (SAS), and therefore $HE = HG$.
17. Therefore, $EFGH$ is a rhombus (a parallelogram with two equal adjacent sides is a rhombus).
18. $m\angle AHD = 90°$ (diagonals of a square are perpendicular to each other).
19. Therefore, a rotation of 90° maps $\overline{DH}$ onto $\overline{AH}$, and thus triangle HDG onto EAH. Thus, $m\angle EHG$ must also be 90°. (Or, alternatively, $m\angle AHD = 90° = m\angle AHE + m\angle EHD$. But from congruency, $m\angle AHE = m\angle DHG$, and therefore $m\angle DHG + m\angle EHD = \angle EHG$.)
20. Therefore, $EFGH$ is a square (a rhombus with a right angle is a square).

In the above explanation (proof), a number of properties are used that students may have encountered previously, but not yet logically explained (proved). This should not present a problem if they later revisit these properties and logically establish them.

Although the above proof uses an elegant argument, some students may find it easier to simply repeat the same argument about corresponding pairs of congruent triangles at vertices B, C, and D. This implies that all four sides are equal (a rhombus), but since the one right angle is already proved, it follows that the quadrilateral must be a square.

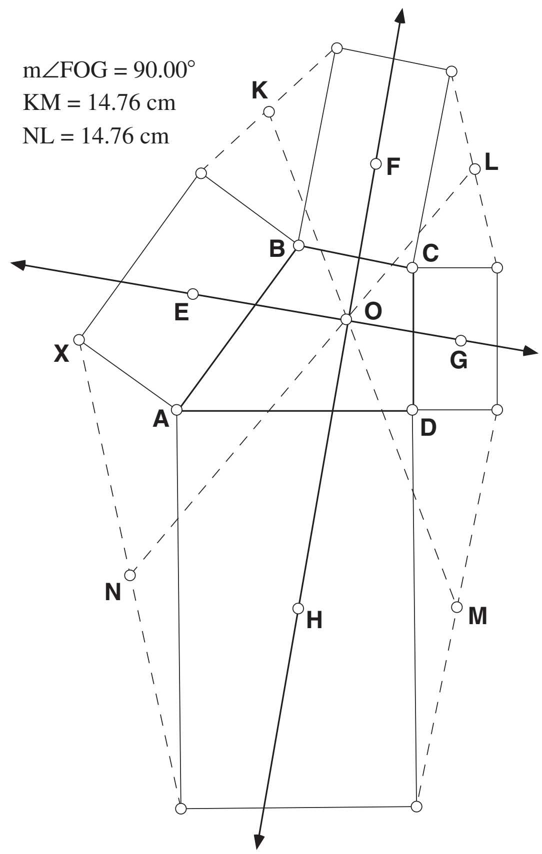
Further Exploration

1. The result still holds if the squares are constructed inwardly, and exactly the same argument applies, except that both $m\angle HAE$ and $m\angle HDG$ are then equal to $m\angle D = 90°$.
2. The centers of the squares on the sides of an isosceles trapezoid form a kite. This follows directly from symmetry; that is, the axis of symmetry of the isosceles trapezoid is also the axis of symmetry of the formed quadrilateral that passes through one pair of opposite vertices (which implies that it is a kite).
3. The centers of the squares on the sides of a kite form an isosceles trapezoid. This also follows directly from symmetry, as in the preceding argument.

Generalizing

You may wish to encourage students to investigate/explain what would happen if they constructed squares on the sides of any quadrilateral. In general, the diagonals of $EFGH$ are equal and perpendicular ($\overline{EG} \perp \overline{HF}$) in any quadrilateral (see Yaglom 1962 , 39 or Kelly 1966).

The latter result, known as van Aubel's theorem, can be further generalized for similar rectangles and rhombuses on the sides as shown below (different proofs are given in de Villiers 1997 and 1998a). In the first figure, $\overline{EG}$ is always perpendicular to $\overline{FH}$. Also, $\overline{KM}$ is congruent to $\overline{LN}$ where K, L, M, and N are the midpoints of the line segments joining adjacent vertices of the similar rectangles as shown. A dynamic sketch is provided in **Aubel 1.gsp.**

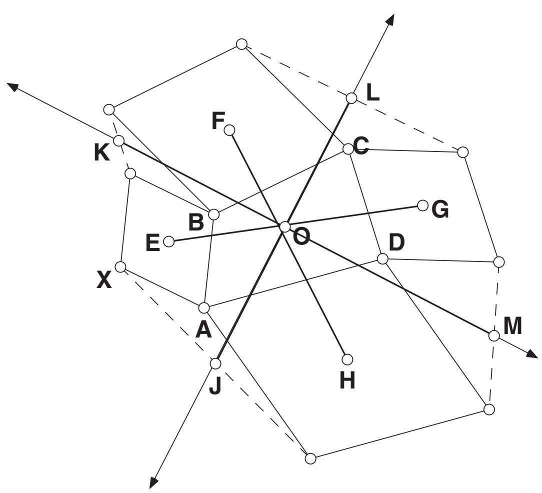


In the second figure, $\overline{EG}$ is always congruent to $\overline{FH}$. Also, $\overline{KM}$ is perpendicular to $\overline{LN}$, where *K, L, M,* and *N* are the midpoints of the line segments joining adjacent vertices of the similar rhombuses as shown. A dynamic sketch is provided in **Aubel 2.gsp.** The "intersection" of these two results therefore yields van Aubel's theorem.

Two interesting special cases are obtained by constructing these similar rectangles and rhombuses on the sides of a parallelogram. In the first case, a rhombus is obtained, and in the second case, a rectangle. Proofs of these two special cases can be found in de Villiers (1996, 101–102).

All these results also nicely display the angle-side duality mentioned in the Teacher Notes for the Isosceles Trapezoid Midpoints activity, as well as in the Teacher Notes for the Logical Discovery: Circum Quad activity.

These two generalizations involving similar rectangles and rhombuses on the sides of any quadrilateral have since been generalized further to parallelograms, and to points other than the "centers" (see de Villiers 2000). A downloadable copy of this paper, as well as Sketchpad 3 sketches illustrating these generalizations, can be found on the author's Web site at http://mzone.mweb.co.za/residents/profmd.



m∠LOM = 90.000°
EG = 8.402 cm
FH = 8.402 cm

THE FERMAT-TORRICELLI POINT (PAGE 108)

This activity reinforces the function of proof discussed earlier, namely, logical discovery. After proving the results for a right triangle, students focus their attention on whether the arguments are still valid if angle *ABC* is not a right angle. This should make them realize that the result is immediately generalizable to *any* triangle. You can emphasize that this often happens in mathematical research, namely, that in proving some result, we find on reflection that some conditions were never used in the proof (i.e., were unnecessary) and that the result can therefore be generalized. The reason for starting with the right triangle is therefore to specifically illustrate this *discovery* function of proof.

Prerequisites: Knowledge of the properties of convex cyclic quadrilaterals (quadrilaterals that can be inscribed in a circle). Specifically, students should know that a convex quadrilateral is cyclic if and only if a pair of its opposite angles are supplementary. These properties have been discovered and proved in two earlier activities: Cyclic Quadrilateral and Cyclic Quadrilateral Converse. Also, students should be familiar with the SAS method of proving a pair of triangles congruent.

Sketch: Fermat 1.gsp. Additional sketches are **Fermat 2.gsp, Fermat 3.gsp,** and **Fermat 4.gsp.**

CONJECTURE

1. The "outer" triangles are all equilateral. If students are uncertain, encourage them to measure the sides or angles.
2. The lines *DC, EA,* and *FB* are concurrent.
3. The line segments *DC, EA,* and *FB* are equal in length.
4. The triangles lie inward.
5. Both results are still true.

CHALLENGE This gives students a first try at writing a proof for their conjectures.

VERIFYING

6. Triangle *DBC* maps onto triangle *ABE* (and they are therefore congruent).