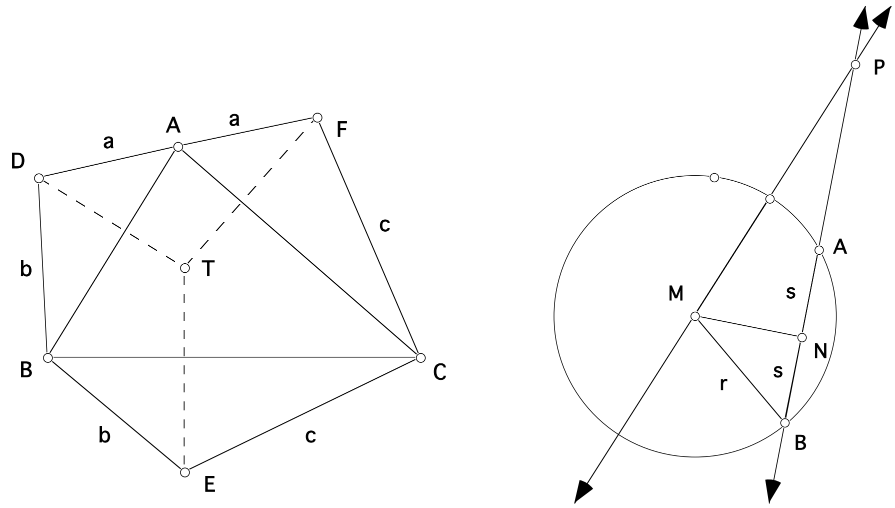

Figure 4.70

However, to prove this result with classical methods we first need the following result.

Lemma

If a line l cuts a circle (M, r) in the points A and B then for every point P of l we have $PA.PB = PM^2 - r^2$.

Proof

Consider Figure 4.70b. MN is perpendicular to chord AB and divides the chord into two equal parts *s*. Applying Pythagoras twice we have:

$$PM^2 = MN^2 + PN^2$$

$$r^2 = MN^2 + s^2$$

$$\Rightarrow PM^2 - r^2 = PN^2 - s^2 = (PN - s)(PN + s) = PA.PB$$

The number $PM^2 - r^2$ is called the *power* of a point P in relation to the circle (M, r). The power of P is negative, zero or positive depending on whether P lies inside, on or outside the circle. If we have two circles $(M1, r1)$ and $(M2, r2)$ intersecting in A and B, then for every point B on the line AB the following holds: $PM_1^2 - r_1^2 = PM_2^2 - r_2^2$. In other words, every point P has equal powers in relation to both circles. The converse is also true, namely that every point P with equal powers in respect to both circles must lie on the line AB. That is why line AB is called the *power line* of the two intersecting circles.

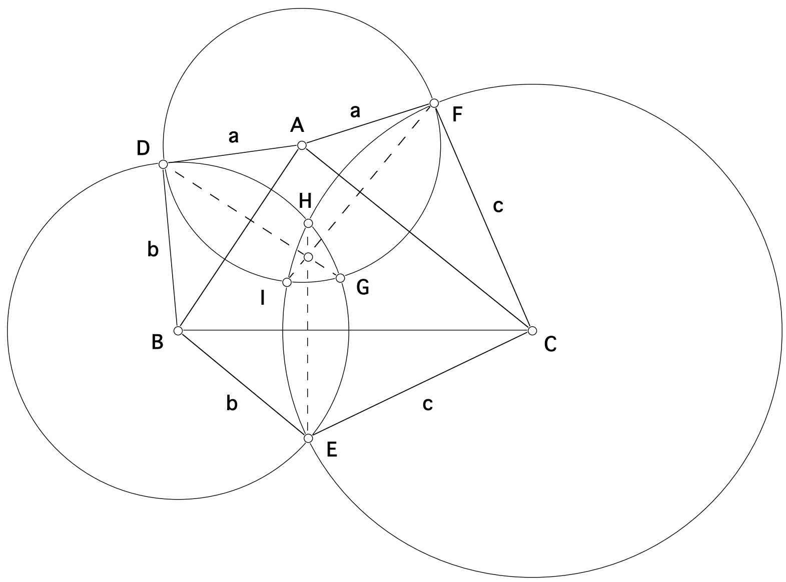

Figure 4.71

Now consider Figure 4.71. Since the three circles each intersect with the other two, we obtain three power lines DG, EH and FI. The intersection point T of say, power lines DG and EH, has equal powers in relation to all three circles and must therefore lie on the third power line FI.

The reader will note that the structure of the above reasoning is similar to loci (symmetry) proofs that the perpendicular or angle bisectors of a triangle are concurrent. (Eg. two perpendicular bisectors meet in a point equidistant from all three vertices and must therefore lie on the third perpendicular bisector. Similarly, two angle bisectors meet in a point equidistant from all three sides and must therefore lie on the third angle bisector).

From the above argument, one can also immediately see that the result would also be valid if the triangles were constructed inwardly. The intersection point of the three power lines is called the *power point* of the three circles.