Hence, the two given equations are equivalent to the one equation $z^4 = 1 + i$ in the complex unknown z. Now write z and $1 + i$ in polar form:

$$z = r(\cos\theta + i\sin\theta), \qquad 1 + i = \sqrt{2}\left(\cos(\pi/4) + i\sin(\pi/4)\right).$$

Using the formula $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\ \sin n\theta$ (de Moivre's theorem), our equation becomes

$$r^4\left(\cos 4\theta + i\sin 4\theta\right) = \sqrt{2}\left(\cos(\pi/4) + i\sin(\pi/4)\right).$$

We get a solution by taking $r = \sqrt[4]{\sqrt{2}}$, $4\theta = \pi/4$, or equivalently, $r = \sqrt[8]{2}$, $\theta = \pi/16$, and therefore $x = \sqrt[8]{2}\cos(\pi/16)$, $y = \sqrt[8]{2}\sin(\pi/16)$. Using half-angle formulas, this solution can be written as shown above.

Comments. There are exactly three other solutions, obtained by taking

$$4\theta = \pi/4 + 2\pi, \qquad 4\theta = \pi/4 + 4\pi, \qquad 4\theta = \pi/4 + 6\pi.$$

For a more elementary (but laborious) approach, rewrite the equations as

$$(x^2 - y^2)^2 - 4x^2y^2 = 1,$$
$$4xy(x^2 - y^2) = 1.$$

Then let $xy = u$, $x^2 - y^2 = v$, so the system becomes $v^2 - 4u^2 = 1$, $4uv = 1$. Although this reduces to a quadratic equation in u^2, considerable computation is needed to eventually recover x and y.

Problem 45

Call a convex pentagon "parallel" if each diagonal is parallel to the side with which it does not have a vertex in common. That is, $ABCDE$ is parallel if the diagonal AC is parallel to the side DE and similarly for the other four diagonals. It is easy to see that a regular pentagon is parallel, but is a parallel pentagon necessarily regular?

Answer. No, a parallel pentagon need not be a regular pentagon.

Solution 1. A one-to-one linear transformation of the plane onto itself takes parallel lines to parallel lines. So, start with a regular pentagon, say with vertices

$$\mathbf{v}_k = \begin{pmatrix} \cos(2k\pi/5) \\ \sin(2k\pi/5) \end{pmatrix}, \qquad k = 0, 1, 2, 3, 4,$$

and now simply change the scale on the x- and y-axes. For example, take the specific linear transformation $\mathbf{A} = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. This stretches the x-coordinate by a factor of 2 and leaves the y-coordinate unchanged. The resulting figure is a non-regular parallel pentagon.

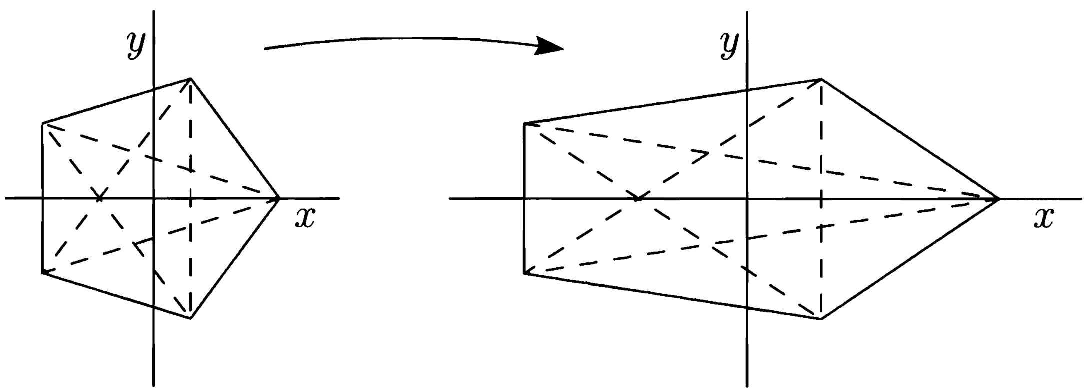

FIGURE 15

Solution 2. Another way to construct a non-regular parallel pentagon is as follows: Start with a square $ABXE$, say of side 1. Extend EX by x units to C and BX by x units to D, where $x > 0$ will be determined later. We will choose x so that $ABCDE$ is a parallel pentagon; obviously it will be non-regular.

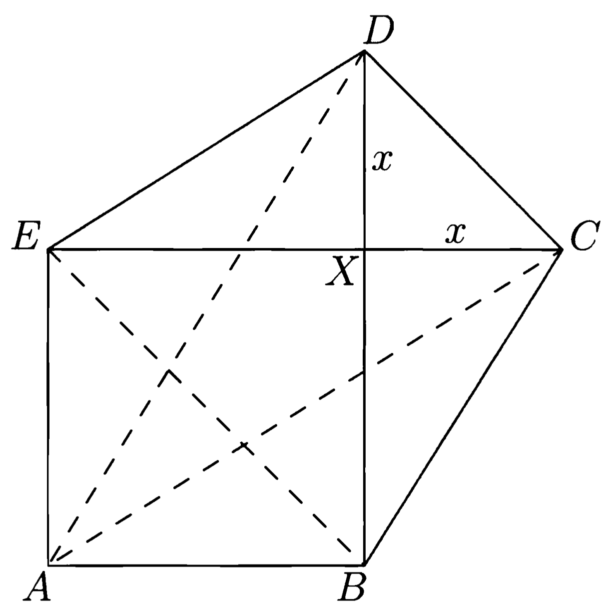

FIGURE 16

Regardless of the value of $x > 0$, $EC \parallel AB$ and $BD \parallel AE$. Also, since $DX = CX = x$, DXC and BXE are both isosceles right triangles, so $BE \parallel CD$. This leaves us to choose x (if possible) so that $AC \parallel DE$ and $AD \parallel BE$. By symmetry about AX, it is enough to find x so that $AC \parallel DE$. For this, it is sufficient to choose x so that triangles DEX and ACE are similar, or equivalently, so that the ratios of corresponding legs of the right triangles are equal to each other. Thus, we want

$$\frac{x}{1} = \frac{DX}{EX} = \frac{AE}{CE} = \frac{1}{1+x}.$$

This holds, for $x > 0$, when $x = (-1 + \sqrt{5})/2$, and thus our construction can be carried out.

Problem 46

Find the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{2n^2 - n} = 1 + \frac{1}{6} + \frac{1}{15} + \frac{1}{28} + \cdots.$$

Answer. The series sums to $2 \ln 2$.

Solution 1. We begin with the partial fraction decomposition

$$\frac{1}{2n^2 - n} = \frac{2}{2n-1} - \frac{1}{n} = \frac{2}{2n-1} - \frac{2}{2n}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{2n^2 - n} = 2\sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n}\right) = 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right).$$

The last step is legitimate because the alternating series on the right is convergent. We now recall the well-known Taylor series expansion

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n},$$

which is valid for $-1 < x \leq 1$. In particular, setting $x = 1$ yields

$$\sum_{n=1}^{\infty} \frac{1}{2n^2 - n} = 2 \ln 2.$$