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Problem Posing Variations on Fermat's Last Theorem

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> "*The fascination of mathematics is fundamentally the same as the fascination of exploration except that the discoveries are made in the realm of ideas rather than in physical space. No doubt the pleasure is greatest when an idea is clarified and isolated after a struggle, but most people have sufficient experience, if only in attempting to solve Christmas puzzles, to enable them to understand the exuberance with which Pythagoras and Archimedes are said to have celebrated their discoveries. It is not possible for our pupils to rediscover the whole of mathematics for themselves, or even those portions of it which seem to the greatest relevance today, but fortunately the pleasure seems to be experienced under guided discovery. It is important that the classroom activities should be carried on with a certain degree of expectancy; new ideas, fresh discoveries, deepened interest are just round the corner waiting to burst in at any moment.*" - Bailey et al (1974: 148)

How is new mathematics discovered or created? Where does this theorem or theory come from? How was it arrived at? What stimulated its conjecture or development?

These are burning questions that are seldom adequately answered for our pupils and students. Traditionally, the teacher just announces a theorem like a magician pulling a rabbit from a hat, leaving pupils (subconsciously) wondering where it came from or how it could have been discovered, and therefore adding to the unsatisfactory mystification of mathematics.

One way of demystifying mathematics is to include suitable *problem posing* or *conjecturing* activities at regular intervals in the classroom and to encourage pupils to formulate their own questions and to investigate them. It should be pointed out to pupils that a good mathematician is not merely a good problem solver, but also a creative problem poser. A mathematician's task is never finished. By looking back at the original problem or its solution, it is almost always possible to pose further problems in relation to it.

One of the problems with traditional teaching is that pupils and students usually do not have any **ownership** over the problems: the problems are usually given to them by either the teacher or the textbook. They are usually not at all involved in the selection of the problems, nor their formulation. In contrast, from personal experience and those of some of my students, it is clear that one is much more motivated if one can formulate one's own questions and investigate them; it gives one a much better sense of ownership and feeling of actually creating new mathematics.

At the heart of making conjectures and problem posing lies the ability to look and ask *questions* from different perspectives. For example, it is a good habit to acquire to ask oneself the following questions whenever one comes across a mathematical problem, result or situation since it may lead to the formulation of new conjectures:

- What if ... is changed?
- What happens if ...?
- What if ... not?

Although the investigation of such questions do not necessarily lead to entirely new and exciting results, one will occasionally be successful if one continually nurtures such a questioning habit. In what follows this "*what-if?*' stategy will be briefly illustrated in relation to the following example.

Fermat's Last Theorem

Great excitement was generated in June 1993 when it became known that Andrew Wiles had managed to prove Fermat's Last Theorem during a conference at Cambridge University. At the time it was front page news in overseas newspapers and even featured on some television news bulletins (sadly not in South Africa). This famous problem which has stubbornly resisted the concerted efforts of mathematicians for three hundred years, can briefly be formulated as follows:

For $n > 2$ there exists no positive integer solutions x, y and z to $x^n + y^n = z^n$.

Although a major problem was encountered with Wiles' proof in November 1993, he has since with the help of Taylor managed to fill the gap (AMS Notices, 42(7), July 1995, 743-746). Interested readers can read more about the interesting history of Fermat's Last Theorem in Singh (1997).

Let us now explore some simple variations on the theorem. For n = 1 or n = 2 the equation $x^n + y^n = z^n$ has infinitely many positive integer solutions for x, y and z, and for n = 0 it has none. But what if we consider negative values of n? What if n = -1? Can we find positive integer solutions to:

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}?$$

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A moment's thought easily produces examples such as:

$$\frac{1}{4}+\frac{1}{4}=\frac{1}{2};$$

$$\frac{1}{6}+\frac{1}{6}=\frac{1}{3}.$$

In these examples $x=y$. Can we find examples where $x\neq y$? From algebra we have:

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{z}$$

$$\Leftrightarrow \frac{x+y}{xy}=\frac{1}{z}$$

$$\Leftrightarrow \frac{xy}{x+y}=z$$

This means that we must look for two positive integers x and y when added together divides without remainder into their product. A quick search easily provides 4 and 12, since their sum, 16, divides exactly into their product, 48. Therefore:

$$\frac{1}{4}+\frac{1}{12}=\frac{1}{3}.$$

What if n = -2? Can we find positive integer solutions to:

$$\frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{z^2}?$$

From algebra we have that this equation is equivalent to:

$$x^2+y^2=\left(\frac{xy}{z}\right)^2.$$

So basically, the problem is reduced to finding a Pythagorean triplet x, y and a so that $x^2+y^2=a^2$ with $a=xy/z$. For z to be an integer, xy/a must be an integer and therefore a must be a factor of xy. Consider a Pythagorean triplet multiplied through by a factor k^2:

$$x^2+y^2=a^2$$

$$\Leftrightarrow (kx)^2+(ky)^2=(ka)^2.$$

Now consider the quotient:

$$\frac{(kx)(ky)}{ka}=\frac{kxy}{a}.$$

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Clearly the product of kx and ky would be exactly divisible by ka if we choose k to be divisible by a For example, starting from the well-known Pythagorean triplet $3^2 + 4^2 = 5^2$ we can choose k = 5 to obtain $15^2 + 20^2 = 25^2$. Therefore $z = \frac{15\times 20}{25}$ = 12 and we obtain the following equality:

$$\frac{1}{15^2} + \frac{1}{20^2} = \frac{1}{12^2}.$$

What if n = -3? Can we find positive integer solutions to:

$$\frac{1}{x^3} + \frac{1}{y^3} = \frac{1}{z^3}?$$

As before this equation is equivalent to the following:

$$x^3 + y^3 = \left(\frac{xy}{z}\right)^3.$$

So again the problem is reduced to finding a triplet of positive integers x, y and a so that $x^3 + y^3 = a^3$ and a is a factor of xy. However, since Fermat's Last Theorem has just been proved true, no such triplet exists. We can therefore generalize Fermat's Last Theorem as follows:

For $|n| > 2$ there exists no positive integer solutions x, y and z to $x^n + y^n = z^n$.

Furthermore, it seems possible to formulate and investigate almost endless variations on Fermat's Last Theorem. For example, can we find positive integer solutions for:

(1) $x^n + y^n = z^n + k$ where $k \in \mathrm{Z}$,

(2) $x^n + y^n = z^{n+k}$ where $k \in \mathrm{Z}$,

(3) $x^n + y^{n+1} = z^{n+2}$,

(4) $x^n + y^{n^2} = z^{n^3}$,

(5) $x^{\frac{1}{n}} + y^{\frac{1}{n}} = z^{\frac{1}{n}}$,

(6) $n^x + n^y = n^z$,

(7) $k_1 x^n + k_2 y^n = k_3 z^n$ where $k_i \in \mathrm{Z}$,

(8) $kx^n + (k+1)y^n = (k+2)z^n$ where $k \in \mathrm{Z}$, etc,

as well as for combinations of the above?

In addition, we could consider the differences between the terms on the left rather than their sums, and the possibility of negative integer solutions for all of the above. Although the solution to some of these questions may be quite trivial, it is important to continually ask such "*what-if?*" questions to oneself in order to

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cultivate a healthy problem posing mindset, even if they lead nowhere interesting or significant. Of course, a cynic may say: what's the point of asking such questions if you don't know that they lead anywhere. The point is: you'll never know unless you ask and investigate!

Further Developments

Interestingly, in 1997 Texas millionaire banker Andrew Beal offered a major cash prize to the first person to prove his generalised version of Fermat's Last Theorem, namely, that if exponents m, n and r all greater than 2, then there are no positive integer solutions to x, y and z in $x^m + y^n = z^r$. This conjecture was in all probability formed only by asking a simple "what-if?" question like those above!

In 2001 the Beale prize stands at $25,000 and will increase by $5000 each year for every year the problem remains unsolved, up to a limit of $50,000. Interested readers can read more about it at:
http://www.maa.org/data/devlin/devlin%5F12%5F97.html

Establishing the validity of Fermat's Last Theorem also involved proving parts of the so-called Taniyama-Shimura-Weil conjecture. In 1999, a proof was discovered for the full version of this conjecture. Interested readers can read more about these developments proof at:
http://www.maa.org/data/mathland/mathtrek%5F11%5F22%5F99.html

> "*Mathematics is the only infinite human activity. It is conceivable that humanity could eventually learn everything in physics or biology. But humanity certainly won't ever be able to find out everything in mathematics, because the subject is infinite.*" - Paul Erdös

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