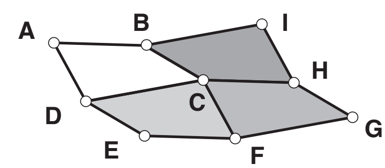
# Quadrilateral Angle Sum

In this activity, you will investigate a quadrilateral tessellation and discover and explain an interesting property of quadrilaterals.

## CONJECTURE




- Open the sketch **Quad Sum.gsp.** Press the buttons to rotate each quadrilateral.
- Reset your sketch. Change the shape of quadrilateral *ABCD* and rotate each quadrilateral again.

**1.** Explain how the four quadrilaterals are related.

**2.** Look at the four angles around vertex *C*. State whether there are any overlaps or gaps between the angles. Also describe the sum of their measures.

**3.** Drag any vertex of quadrilateral *ABCD* to change the quadrilateral. Are your observations from Question 2 still true?

**4.** Carefully compare the angles around vertex *C* with the interior angles of quadrilateral *ABCD*. Measure some angles, if necessary.

**a.** What can you say about $\angle ADC$ and $\angle FCD$? Why?

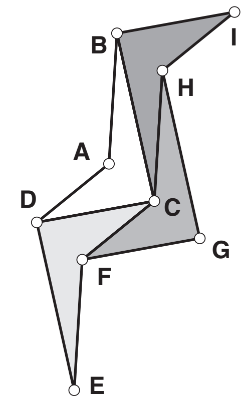
To measure an angle, select three points, using the vertex as the middle selection. Then choose **Angle** from the Measure menu.

**b.** What can you say about $\angle BAD$ and $\angle HCF$? Why?

**c.** What can you say about $\angle CBA$ and $\angle BCH$? Why?

**5.** What can you therefore say about the sum of the interior angles of *ABCD*?

**6.** Drag a vertex until *ABCD* is concave.
Does your conjecture still appear to be true?

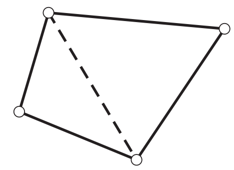



**CHALLENGE** Try to logically explain your conjecture from Question 5 by writing a carefully structured argument based on the preceding exploration. If you get stuck or want some hints, continue reading.

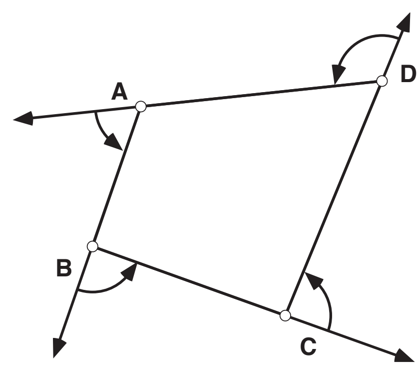
# EXPLAINING

There are several different ways to construct logical explanations as to why the sum of the measures of the interior angles of any convex or concave quadrilateral is 360°. Questions 7 and 8 show two possible ways.

**7.** Reset your sketch. Construct a diagonal that divides the quadrilateral into two triangles. Drag your quadrilateral to make sure this construction holds for both convex and concave cases. Now use what you know about the sum of the measures of the angles of a triangle to explain why the sum of the measures of the interior angles of any quadrilateral is 360°.



**8.** Recall that the sum of the measures of the exterior angles of a polygon is always 360° (as long as none of the sides of the polygon cross). Use this exterior angle sum result to determine the interior angle sum of a quadrilateral.




## Present Your Explanation

Write out one or both of your logical explanations clearly for presentation to the class or to your group. Your summary may be in paper form or electronic form and may include a presentation sketch in Sketchpad. You may want to discuss the summary with your partner or group.

## Further Exploration

Investigate the interior angle sums for convex or concave pentagons, hexagons, and so on. Can you derive a general formula for the interior angle sum of any given convex or concave polygon?

6. Therefore $m\angle ACB + m\angle ABC + m\angle BAC = 180°$.
7. When a transversal cuts across two parallel lines, corresponding and alternate angles formed are equal. The measure of a straight angle is 180°.

### Present Your Explanation

Here, you could discuss different forms of presentation for logical arguments. For example, compare an essay-type with a traditional two-column-type presentation of a logical argument. Students should have some freedom of choice, provided that their presentations are systematic and logical.

### Further Exploration

1. a. $m\angle DBA = m\angle BAC$ (alternate angles). $m\angle EBC = m\angle BCA$ (alternate angles). But $m\angle DBA + m\angle ABC + m\angle CBE = 180°$. (*DBE* is a straight line from construction.) Therefore, $m\angle ACB + m\angle ABC + m\angle BAC = 180°$.

   b. There is hardly any difference between the two explanations, except that the first uses alternate angles and corresponding angles, whereas the second uses only alternate angles. The first explanation also requires the construction of two lines (or rays), whereas the second requires only one line. It could therefore be argued that the second explanation is (slightly) more economical than the first.

## QUADRILATERAL ANGLE SUM (PAGE 37)

**Prerequisites:** Students should be familiar with the sum of the measures of the angles of a triangle. Even better, they have completed the previous activity Triangle Angle Sum. It will also help if they know that the sum of the measures of the exterior angles of a simple closed polygon is 360°. This activity also works well as preparation for the next activity, Crossed Quadrilateral Sum.

**Sketch: Quad Sum.gsp.**

### CONJECTURE

1. All four quadrilaterals, *ABCD, HCBI, FEDC,* and *CHGF,* are congruent to each other.
2. There are no overlaps or gaps between the angles. Since these angles fit around a point, their sum must be 360°.
3. The same observation holds for any convex or concave quadrilateral.
4. a. $m\angle ADC = m\angle FCD$ because a half-turn around the midpoint of $\overline{DC}$ maps these two angles onto each other.

   b. $m\angle BAD = m\angle HCF$ because a translation of quadrilateral *ABCD* to *CHGF* maps these two angles onto each other. Also, students could describe the two rotations in the sketch that map *ABCD* to *CHGF.*

   c. $m\angle CBA = m\angle BCH$ because a half-turn around the midpoint of $\overline{BC}$ maps these two angles onto each other.
5. Since the measure of each of the interior angles of *ABCD* equals the measure of one of the four angles around vertex *C,* it follows that the sum of the measures of the angles of any convex or concave quadrilateral must also be 360°. (Students could also tessellate the quadrilateral around the other vertices.)
6. Students should notice that in the concave case the angles still fit around the same vertex without any overlaps or gaps between them; therefore, the result is also true for concave quadrilaterals.

**CHALLENGE** Here, students are given the opportunity to attempt their own logical explanations.

## EXPLAINING

7. Any convex or concave quadrilateral can be divided into two triangles by drawing an *interior* diagonal. Since the sum of the measures of the angles of any triangle is 180°, the sum of the angles of any convex or concave quad is $2 \times 180° = 360°$.
8. Consider the convex and concave quadrilaterals *ABCD* shown below. In both cases, the sum of the measures of the exterior angles is 360°. Therefore, $m\angle p + m\angle q + m\angle r + m\angle s = 360°$, and the sum of the measures of the interior angles is therefore given by

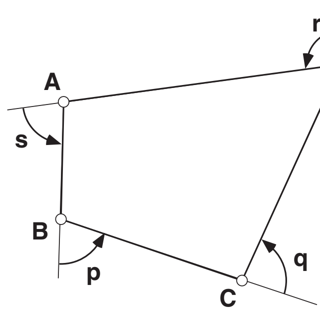
$$(180° - m\angle p) + (180° - m\angle q) + (180° - m\angle r) + (180° - m\angle s)$$

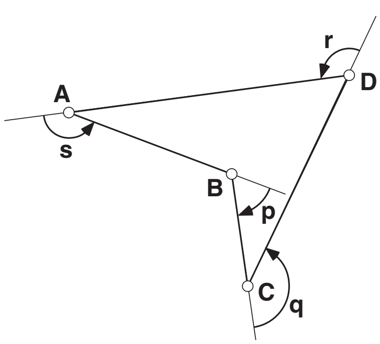
$$= 720° - (m\angle p + m\angle q + m\angle r + m\angle s)$$

$$= 720° - 360° = 360°$$

Note that in the concave case shown, the measure of angle *p* is negative in relation to the other angles, since it has an opposite direction of rotation. The measure of the interior angle at *B* (the reflex angle) is therefore

$$180° - m\angle p = 180° + |m\angle p|$$




## Present Your Explanation

Here, students are given the opportunity to summarize and present their arguments for discussion.

## Further Exploration

One way to arrive at a general formula is to observe that any concave or convex *n*-gon (for $n > 2$) can be divided into $n - 2$ triangles, and therefore the sum of the measure of the interior angles is $(n - 2)180°$.

Another way is to use the fact that the sum of the measures of the exterior angles of any polygon is 360° (assuming positive orientation). The sum of the measures of the interior angles of any concave or convex *n*-gon (for $n > 2$) is therefore given by

$$(n \times 180°) - \text{sum of exterior angles} = (n \times 180°) - 360° = (n - 2)\,180°$$