Quadrilaterals of triangle centres

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Quadrilaterals of triangle centres

MARIA FLAVIA MAMMANA and BIAGIO MICALE

Introduction

Let Q be a convex quadrilateral *ABCD*. We denote by T_A, T_B, T_C, T_D, the four triangles *BCD*, *CDA*, *DAB*, *ABC*, respectively.

The barycentres (or centroids), orthocentres, incentres and circumcentres of such triangles determine other quadrilaterals in their turn that we call the quadrilateral of the barycentres, of the orthocentres, of the incentres and of the circumcentres, respectively. We denote these quadrilaterals by Q_b, Q_o, Q_i, Q_c, respectively.

Interesting properties have been found for quadrilaterals Q_b, Q_o, Q_i in the case when Q is cyclic (that is, Q is inscribable in a circle, whose centre is called circumcentre of Q): Q_b is similar to Q, Q_o is congruent to Q, Q_i is always a rectangle (cf. [1, 2]). As far as Q_c is concerned, it has been proved that if Q is not cyclic, the quadrilateral of the circumcentres relative to Q_c is similar to Q (cf. [1, 3, 4]). Observe that if Q is cyclic, then Q_c is a point.

In this work, other properties of the quadrilaterals Q_b, Q_o, Q_i, Q_c are determined. Here are our main results:

- the quadrilateral of the barycentres, Q_b, relative to any quadrilateral Q, is the correspondent of Q in the inverse homothety, of ratio 1:3, with the centroid of Q at its centre (section 1).
- the quadrilateral of orthocentres, Q_o, relative to a cyclic quadrilateral Q, is the correspondent of Q in the half-turn about the anticentre of Q (section 2).
- if the quadrilateral of the incentres, Q_i, relative to a quadrilateral Q, is a rectangle, then Q is cyclic (section 3).
- the quadrilateral of the circumcentres, Q_c, relative to a non-cyclic quadrilateral Q, is affine to Q (section 4).

We also prove that the quadrilateral, Q_m, determined by the maltitudes of a non-cyclic quadrilateral Q is affine to Q (section 5).

As these results are easy to prove, material represented here can be utilised in research laboratories with students, as showed in [5]. From this point of view a geometric software like *Cabri Géomètre* is very useful to explore geometric figures, to find out new properties and possible demonstration strategies.

At the end of the paper, the following conjecture is indicated, suggested by using Cabri: Let Q be any convex quadrilateral and Q_0 be its quadrilateral of orthocentres: Q_0 has the same area as Q and is affine to Q.

1. *The quadrilateral of the barycentres*

Let A', B', C', D' be the barycentres of the four triangles T_A, T_B, T_C, T_D. We call the quadrilateral Q_b with vertices A', B', C', D', the *quadrilateral of the barycentres relative to* Q.

We recall that the straight lines joining the midpoints of two opposite sides are called bimedians, and that their common point, called the *centroid* of Q, bisects them both.

Also, given a side of Q, the perpendicular line passing through the midpoint of the opposite side is called the *maltitude* relative to the given side. The four maltitudes of Q are concurrent in a point called the *anticentre* of Q, if, and only if, Q is cyclic. If Q is cyclic, the anticentre and the circumcentre are symmetric with respect to the centroid of Q (cf. [6, 7, 8]).

Theorem 1

The quadrilateral of the barycentres, Q_b, is directly similar to Q. More precisely, Q_b is the correspondent of Q in the inverse homothety, of ratio 1:3, with the centroid of Q at the centre.

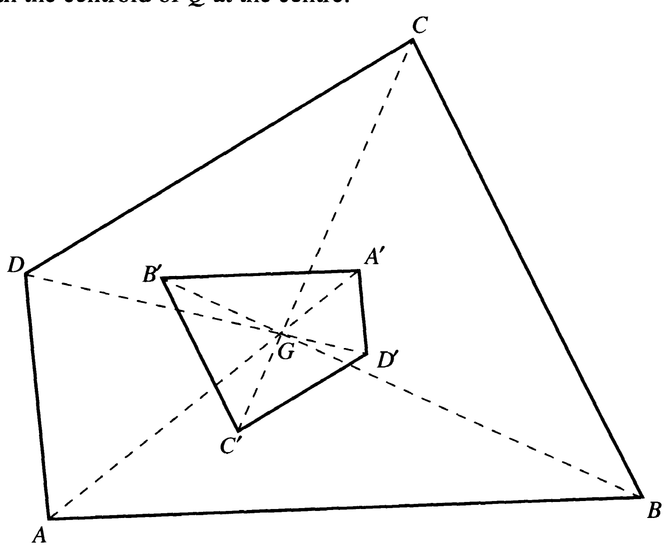

FIGURE 1

The proof is simple using vectors (cf. [1]). If P is any point and G is the centroid of Q (Figure 1), then $\overrightarrow{PG} = \frac{1}{4}\left(\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD}\right)$ and $\overrightarrow{PA'} = \frac{1}{3}\left(\overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD}\right)$, hence

$$\overrightarrow{A'G} = \overrightarrow{PG} - \overrightarrow{PA'} = \tfrac{1}{12}\left(3\overrightarrow{PA} - \overrightarrow{PB} - \overrightarrow{PC} - \overrightarrow{PD}\right)$$

$$= \tfrac{1}{3}\left(\overrightarrow{PA} - \tfrac{1}{4}\left(\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD}\right)\right) = \tfrac{1}{3}\left(\overrightarrow{PA} - \overrightarrow{PG}\right) = \tfrac{1}{3}\overrightarrow{GA},$$

which suffices.

Also, any quadrilateral Q' can be considered as a quadrilateral of the barycentres relative to a particular quadrilateral, more precisely the quadrilateral correspondent of Q' in the inverse homothety of ratio 3 having the centroid of Q' as its centre.

Let us observe that if Q is cyclic, then also Q_b is cyclic. Also, the anticentres of Q and Q_b correspond in the homothety φ as well as Q and Q_b circumcentres. Since in a cyclic quadrilateral, the anticentre and the circumcentre are correspondent in the half-turn about the centroid, we have that the anticentres and the circumcentres of Q and Q_b as well as their common centroid G are collinear.

2. *The quadrilateral of the orthocentres*

Let A', B', C', D' be the orthocentres of triangles T_A, T_B, T_C, T_D respectively. We call the *quadrilateral of the orthocentres relative to* Q the quadrilateral Q_o with vertices A', B', C', D'.

Theorem 2

If Q is a cyclic quadrilateral, then the quadrilateral of the orthocentres Q_O is directly congruent to Q. More precisely, Q_O is the correspondent of Q in the half-turn about the anticentre of Q.

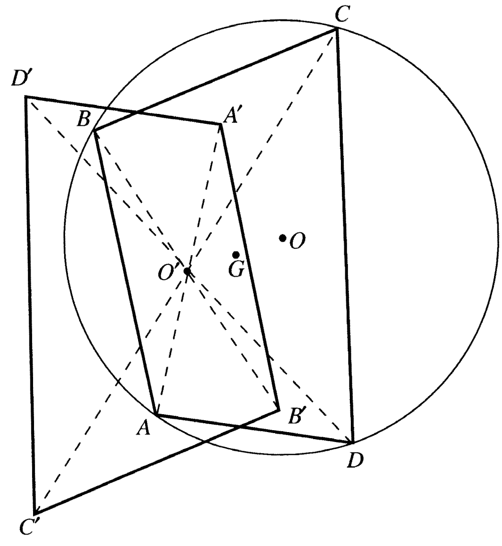

FIGURE 2

Let us suppose Q to be cyclic. We denote with O the circumcentre, with G the centroid, with O' the anticentre of Q (Figure 2).

In the triangle T_A, $\overrightarrow{OA'} = \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}$. Since O and O' are symmetric with respect to G, then $\frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}\right) = \overrightarrow{OO'}$. It follows that O' is the midpoint of segment $\overrightarrow{AA'}$, which suffices.

From Theorem 2 it follows that any cyclic quadrilateral Q' can be considered as a quadrilateral of the orthocentres relative to a particular quadrilateral, more precisely the quadrilateral correspondent of Q' in the half-turn about the anticentre of Q'.

Besides, we observe that Q and Q_o have the same anticentre O' and that their circumcentres, as well as their centroids, are correspondent in the half-turn about O'. It follows that circumcentres and centroids of Q and Q_o and their common anticentre O' are collinear.

Considering what was observed in section 1, we can state that in a cyclic quadrilateral Q, centroids, circumcentres and anticentres of Q, Q_b and Q_o are all collinear.

3. *The quadrilateral of the incentres*

Let A', B', C', D' be the incentres of triangles T_A, T_B, T_C, T_D respectively. We call the quadrilateral Q_i with vertices A', B', C', D' the *quadrilateral of the incentres relative to Q.*

Theorem 3

The quadrilateral Q_i of the incentres relative to a quadrilateral Q is a rectangle if, and only if, Q is cyclic.

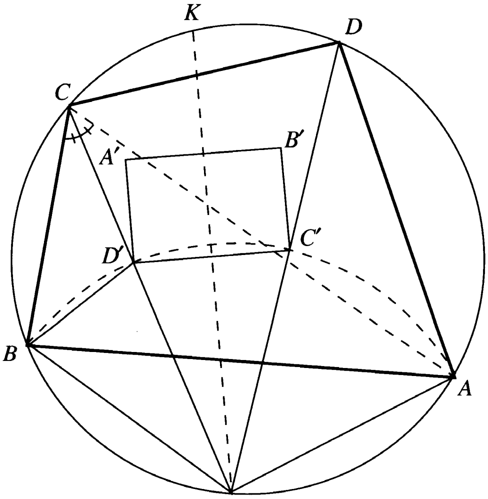

FIGURE 3a

It is known that if Q is a cyclic quadrilateral, then Q_i is a rectangle (cf. [1, 2]). We now give a proof of this.

Consider the circumcircle of Q (Figure 3a). The straight lines CD' and DC' pass through the midpoint H of the arc AB.

Now $BH = AH$. Moreover, $\angle HBA = \angle HCA = \angle BCH$ and $\angle D'BC = \angle ABD'$, hence

$$\angle HBD' = \angle HBA + \angle ABD' = \angle BCH + \angle CBD' = \angle HD'B.$$

So the triangle BHD' is isosceles and $BH = D'H$. Similarly, $AH = C'H$. Hence the points B, D', C', A belong to the circle through B with centre H.

If K is the midpoint of the arc CD, then HK, being the internal bisector of $\angle D'HC'$, is the perpendicular bisector of $C'D'$. For the same reason, HK is also the perpendicular bisector of $A'B'$. It follows that Q_i is an isosceles trapezium. The same reasoning also shows that the straight line joining the midpoints of the arcs CB and AD is the common perpendicular bisector of $A'D'$ and $B'C'$. Hence Q_i is a rectangle.

We prove now that if Q is a non-cyclic quadrilateral, then Q_i is not a rectangle.

The circle passing through B, C, D does not contain A. Let us suppose that the straight line AC meets the circle at A_1.

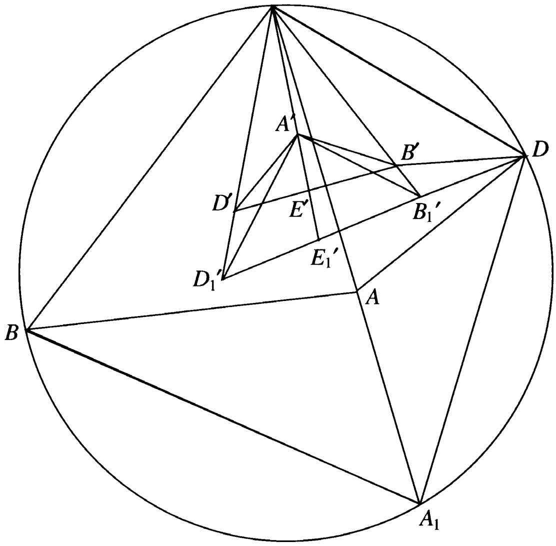

FIGURE 3b

Let A', D', B' be the incentres of the triangles T_A T_D, T_B; let D'_1, B'_1 be the incentres of the triangles A_1BC, CDA_1; let E' and E_1' be the points in which the straight line CA' meets $B'D'$ and $B_1'D_1'$ respectively. Let us prove that the angle $\angle B'A'D'$ is not right.

Let us suppose that A is inside the circle (Figure 3b). Then $\angle ADC < \angle A_1DC$. Since DB' and DB_1' are internal bisectors of the angles $\angle ADC$ and $\angle A_1DC$ respectively, we have $\angle CDB' = \angle CDB_1'$. It follows that B' is an interior point of the segment CB_1'. Similarly, D' is an interior point of the segment CD_1'.

Then $\angle CA'B' < \angle CA'B_1'$ and their supplements $\angle B'A'E'$ and $\angle B_1A'E_1'$, are such that $\angle B'A'E' > \angle B_1'A'E_1'$. Similarly $\angle D'A'E' > \angle D_1'A'E_1'$. It follows that $\angle B'A'D' > \angle B_1'A'D_1'$. Since A_1BCD is a cyclic quadrilateral, $\angle B_1'A'D_1'$ is right, so $\angle B'A'D'$ is obtuse. In a similar way we can prove that if A is outside the circle passing through B, C, D, then $\angle B'A'D'$ is acute.

Therefore the quadrilateral of incentres relative to $ABCD$ is not a rectangle.

Let us consider now any quadrilateral Q and its quadrilateral of incentres Q_i.

It is easy to prove that if a point P is centre of symmetry for Q, then P is centre of symmetry for Q_i, too; and if a straight line r is the axis of symmetry passing through the vertices (respectively, not passing through vertices) for Q, then r is the axis of symmetry passing through vertices (respectively, not passing through vertices) for Q_i, too.

It follows that if Q is a parallelogram, Q_i is also a parallelogram; if Q is a kite, Q_i is also a kite; if Q is a rhombus, Q_i is also a rhombus.

We also observe that if Q is a cyclic kite, Q_i is a square (because it is a kite and a rectangle at the same time).

By using *Cabri* it appears that kites are not the only quadrilaterals Q such that Q_i is a square. We propose then the following open problem: determine all (cyclic) quadrilaterals Q such that Q_i is a square.

4. *The quadrilateral of the circumcentres*

Let A', B', C', D' be the circumcentres of triangles T_A, T_B, T_C, T_D respectively. We call the quadrilateral Q_c with vertices A', B', C', D' the *quadrilateral of the circumcentres relative to Q*. We have already observed that if Q is cyclic, then Q_c is reduced to a point, so we consider here only the case in which Q is not cyclic.

Let us make some preliminary remarks.

Let M be the common point of its diagonals AC and BD. For the two ratios $\frac{AM}{MC}$ and $\frac{CM}{MA}$, let h be the one not greater than 1. Also for the two ratios $\frac{BM}{MD}$ and $\frac{DM}{MB}$, let k be the one not greater than 1. The pair $\{h, k\}$ is called the *characteristic* of Q ([9]). Q has characteristic $\{h, h\}$ if, and only if, it is a trapezium; Q has characteristic $\{1, 1\}$ if, and only if, it is a parallelogram. In [9] it was proved that two quadrilaterals are affine if and only if they have the same characteristic.

Theorem 4

The quadrilateral Q_c of the circumcentres relative to a (non-cyclic) quadrilateral Q is affine to Q.

A' is the centre of the circle passing through B, C, D and C' is the centre of the circle passing through A, B, D, so the straight line $A'C'$ is perpendicular to the radical axis, BD, of such circles (Figure 4a). Similarly, the straight line $B'D'$ is perpendicular to the line AC. Further, the lines $A'B'$, $B'C'$, $C'D'$, $A'D'$ are perpendicular to the lines CD, AD, BA, CB respectively.

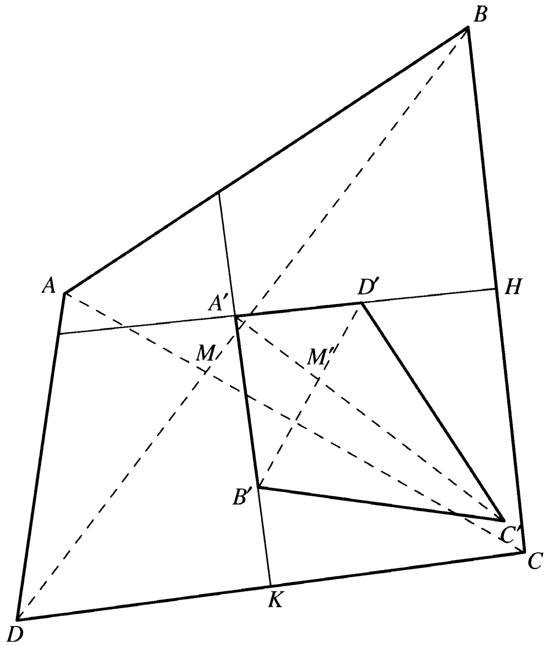

FIGURE 4a

It follows that the triangles ABM and $C'D'M'$, BCM and $A'D'M'$, CDM and $A'B'M'$ are similar. Therefore we have:

$$(1)\ \frac{AM}{BM} = \frac{M'D'}{M'C'}; \qquad (2)\ \frac{BM}{MC} = \frac{A'M'}{M'D'}; \qquad (3)\ \frac{MC}{MD} = \frac{B'M'}{A'M'}.$$

By multiplying member by member (1) and (2) and then (2) and (3) we have:

$$\frac{AM}{MC} = \frac{A'M'}{M'C'}; \qquad \frac{BM}{MD} = \frac{B'M'}{M'D'}.$$

Thus, the quadrilaterals Q and Q_c have the same characteristic and are affine.

From Theorem 4, it follows that if Q is a trapezium, then Q_c is a trapezium and that if Q is a parallelogram, then Q_c is a parallelogram.

The proof of Theorem 4 shows that the two quadrilateral Q and Q_c, besides being affine, are such that the diagonals of Q are perpendicular to that of Q_c so the angles formed by the diagonals of Q are congruent to those formed by the diagonals of Q_c. Moreover, it can be easily proved that the angles of Q are supplementary of the angles of Q_c. For example, the angles in H and in K of the quadrilateral $A'HCK$ are right, so the angles in A' and in C are supplementary, therefore the angle in A' of Q_c is the supplement of the angle in C of Q. It follows that Q_c is not cyclic. In fact, if Q_c was cyclic, its opposite angles would be supplementary, so also the opposite angles of Q would be supplementary and so Q would be cyclic.

Since Q_c is not cyclic, we can consider the quadrilateral of the circumcentres, $(Q_c)_c$, relative to Q_c, with vertices A'', B'', C'', D'' (Figure 4b). Let M'' be the common point of the diagonals $A''C''$ and $B''D''$.

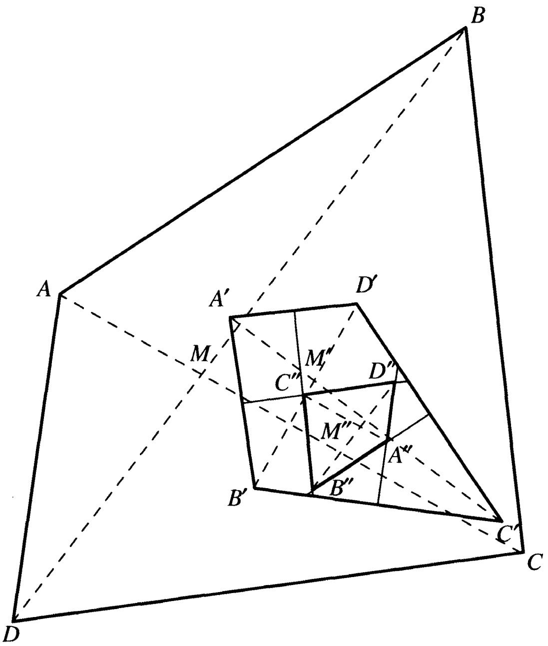

FIGURE 4b

Theorem 5

The quadrilateral $(Q_c)_c$ is homothetic to Q.

In fact, Q and $(Q_c)_c$ have their sides and diagonals respectively parallel because they are perpendicular to the sides and diagonals of Q_c respectively. So Q and $(Q_c)_c$ have their angles respectively congruent. Further, triangles ABD and $A''B''D''$ are similar as well as triangles BCD and $B''C''D''$, so:

$$\frac{AB}{A''B''} = \frac{AD}{A''D''} = \frac{BD}{B''D''}; \qquad \frac{BC}{B''C''} = \frac{CD}{C''D''} = \frac{BD}{B''D''}.$$

We observe that quadrilaterals Q and $(Q_c)_c$, besides being similar, are such that the corresponding diagonals and sides are parallel, so they are also homothetic.

5. *Quadrilateral of maltitudes*

Finally, let us consider the maltitudes of the quadrilateral Q and the quadrilateral Q_m with vertices $\overline{A'}$, $\overline{B'}$, $\overline{C'}$, $\overline{D'}$ that they determine (Figure 5). More precisely, $\overline{A'}$ is the common point to the maltitudes M_1H_1 and M_2H_2 relative to sides BC and CD respectively; $\overline{B'}$ is the common point to the maltitudes M_2H_2 and M_3H_3; $\overline{C'}$ is the common point to the maltitudes M_3H_3 and M_4H_4; $\overline{D'}$ is the common point to the maltitudes M_4H_4 and M_1H_1.

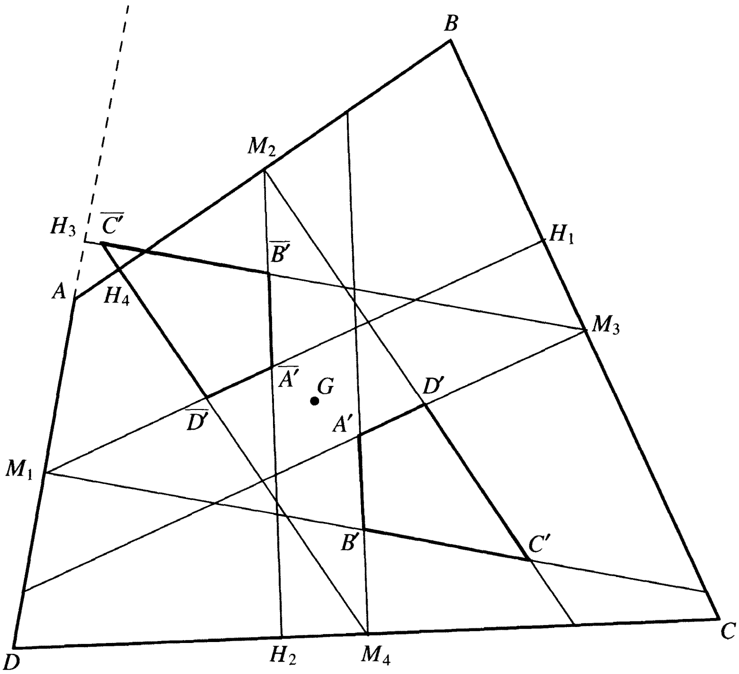

FIGURE 5

Theorem 6

The quadrilateral Q_m determined by the maltitudes of a non-cyclic quadrilateral Q is the correspondent of the quadrilateral of the circumcentres Q_c relative to Q in the half-turn about the centroid of Q.

In fact, M_1 and M_3 are symmetric with respect to the centroid G of Q, as well as M_2 and M_4, therefore the half-turn about G transforms the maltitude M_1H_1 into the parallel to M_1H_1 passing through M_3, that is in the perpendicular bisector of the side BC. Thus, the maltitudes of Q are transformed into the perpendicular bisectors of Q so that the quadrilaterals Q_m and Q_c are symmetric with respect to G.

From Theorems 4 and 6 follows:

Theorem 7

The quadrilateral Q_m is affine to Q.

Conjecture: Let Q be any convex quadrilateral and Q_o be its quadrilateral of orthocentres: Q_o has the same area of Q and is affine to Q.

The conjecture arises from the fact that all the proofs that we made by using *Cabri* indicate that Q and Q_o have the same area and moreover that the characteristics of Q and Q_o are the same. However, we have been unable to prove the conjecture.

Further reading

Whilst not referred to explicitly in this paper readers may wish to look at these books:

H. S. M. Coxeter and S. L. Greitzer, *Geometry revisited*, Washington, DC: Amer. Math. Assoc. (1967).

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