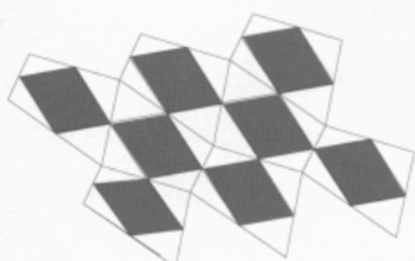
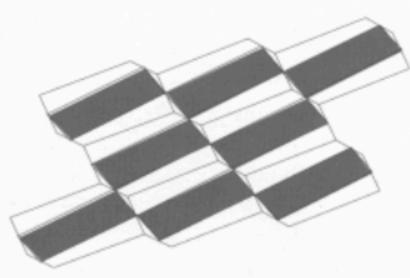
Quadrilaterals and Parallelo-hexagons

by **Paul Stephenson**

In 'Some Hexagon Area Ratios: Problem Solving by Related Example' (*Mathematics in School*, **34**, 1, January 2010), Michael de Villiers shows us yet again that the mine of elementary geometry is inexhaustible.

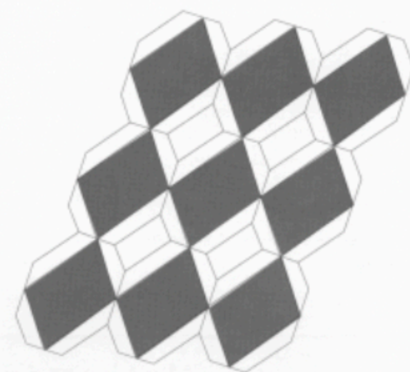
Readers should look back at p.21 of that issue, where he reveals a property shared by the general quadrilateral and the 6-sided zonagon or 'parallelo-hexagon': a parallelogram which joins midpoints of sides is half the area of the figure. Another is that both shapes tessellate. The result is a checkerboard pattern made by the two congruent parallelograms. In the case of the quadrilateral (his Fig. 2), one is situated like EFGH, the other like HGG'H'. In the case of the parallelo-hexagon (Fig. 3), one is situated like GHIJ, the other like HH'I'I. (You will appreciate that in each case the parallelogram pair forms a unit cell of the tiling.) Here are my tiled versions of the respective figures:





We can think of these as regular tilings distorted by a series of parallel-preserving transformations.

De Villiers finds that the 'halving' property does not hold for the parallelo-*octagon*. We can confirm this for the parallelogram which joins midpoints of alternate sides. Construct a distorted version of the semiregular tiling 4.8^2, our parallelogram connecting sides of bordering octagons:



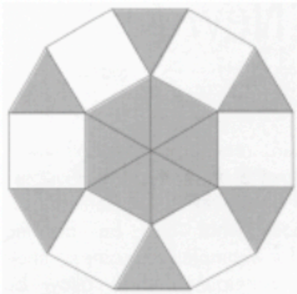
We see that this parallelogram is congruent to one containing both the remaining pieces of the octagon and the distorted square. Since the size of that is not specified, the area of the parallelogram bears no relation to that of the octagon which contains it.

Your KS3 students (including Y7) should find these constructions satisfying. They may also like to find how many different 'midpoint' parallelograms the general parallelo-*n*-gon contains. My answer is

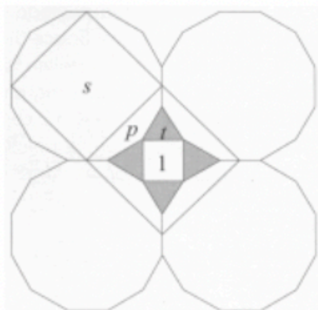
$$\binom{\frac{1}{2}n}{2},$$

viz. 1 for the parallelogram itself, 3 for the parallelo-hexagon, 6 for the parallelo-octagon, ... Encourage them to make their count in several different ways in case the reasoning behind one of them is flawed. (It's so easy to underestimate by missing cases and overestimate by double-counting.)

We can use a tiling to tell us the relative area of one particular midpoint parallelogram in one particular parallelo-*n*-gon: the square (of area *s*) formed by joining the midpoints of every 3rd side of a regular dodecagon (of area *d*), isolating 4 congruent irregular pentagons (each of area *p*). This is possible because we have an extra piece of information: the regular dodecagon dissects into 6 unit squares and 12 equilateral triangles of unit side:



Assume for the moment we don't know the area of the second in terms of the first and call it t. We use the tiling 3.12^2 / 3.4.3.12 :



We see that we can express s in two ways:

1. $s = d - 4p = (6 + 12t) - 4p,$
2. $s = 1 + 4t + 4p,$

algebra in the Geoff Giles tradition.
We solve the equations to give $\frac{s}{d} = \frac{7+16t}{12+24t}$.

If we import the information that $t = \frac{\sqrt{3}}{4}$, we have $\frac{s}{d} = \frac{2+\sqrt{3}}{6}$.

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