A Quarter-Circle Investigation, Explanation and Generalization

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An interesting geometric conservation problem is presented. Here proof is presented in a 'proof without words' style, with the aim of developing the reader's visual proof ability. The study of the task and its expansion is accompanied by a dynamic sketch to highlight the conservation property.

INTRODUCTION

Here is a little investigative task suitable for high school geometry learners of any level. It requires only basic knowledge of the properties of lines, circles, some circle geometry theorems, and similarity. While it is not essential, for enhanced learning, use of dynamic geometry for the investigation is strongly recommended. If time allows, a teacher might give learners instructions on how to create their own dynamic geometry sketches, or the teacher may wish to demonstrate the construction in front of the class with a computer connected to a data projector.

However, for convenience, and to save valuable teaching and learning time, the reader is provided with ready-made sketches for exploration or demonstration at: http://dynamicmathematicslearning.com/quarter-circle-investigation.html

TASK 1

A quarter of a circle with center O is given, and AO and BO are perpendicular radii. The point C is any point on the arc of the circle BC.

The segment AD is perpendicular to chord BC extended ($AD \perp CD$), as shown in Figure 1

1) Determine the size of $\sphericalangle BCA$.
2) Prove that right ΔCDA is isosceles (i.e. $DC = DA$).

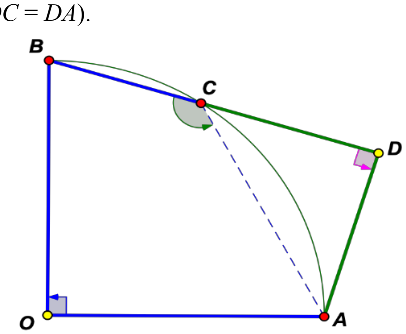

Figure 1

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While the task is elementary enough for most learners to probably solve quite easily without the visualisation aid of dynamic geometry, the software is likely to create additional conceptual awareness and appreciation that for each position of the point C as it is dragged along arc BA, the value of $\sphericalangle BCA$ remains constant (invariant), and that AD remains equal to CD. However, while the software empirically confirms these observations, it provides no explanation (proof) of why the results are true.

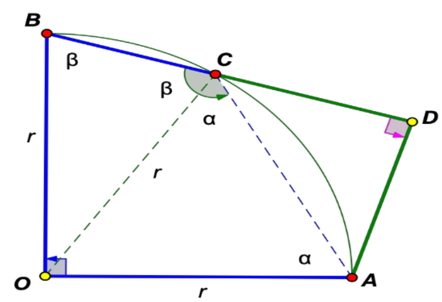

Figure 2

PROOF (EXPLANATION) OF TASK 1

Construct auxiliary line OC. Consider Figure 2.

$\Delta OBC \Rightarrow OB = OC = r \Rightarrow \sphericalangle OBC = \sphericalangle OCB = \beta$
$\Delta OCA \Rightarrow OC = OA = r \Rightarrow \sphericalangle OCA = \sphericalangle OAC = \alpha$
Quadrilateral $OABC \Rightarrow 2\alpha + 2\beta + \sphericalangle AOB = 360° \Rightarrow 2\alpha + 2\beta = 270° \Rightarrow$
$\alpha + \beta = 135°$.
Therefore $\sphericalangle BCA = 135° \Rightarrow \sphericalangle ACD = 45° \Rightarrow \sphericalangle CAD = 45° \Rightarrow DA = DC$.

Also note that the reason that $\sphericalangle BCA$ remains constant as C is dragged along arc BA, is that all these angles $\sphericalangle BCA$ on the circumference are subtended by same chord AB, and hence are all equal.

GENERALIZE

Looking back at the result, it is immediately evident that $OABC$ is a cyclic quadrilateral since it has a pair of opposite right angles at O and D. Further reflection on the proof now reveals that the result (DC=DA) generalizes for any $\sphericalangle AOB$ between (0°, 180°) as long as the extension of BC meets the circumcircle of ΔAOB at D, or phrased differently, as long as $OADB$ is cyclic (i.e. $\sphericalangle AOB$ + $\sphericalangle BDA$ = 180°) as shown in

Figure 3. This generalization, resulting from further reflection on the preceding task, once again illustrates the value of the so-called ‘discovery’ function of proof (De Villiers, 1990; 2003).

PROOF (EXPLANATION) OF GENERALIZATION

Consider Figure 3. The first two lines of the proof remain identical to that of Task 1. We pick up the proof from the third line above:

Quadrilateral $OABC \Rightarrow 2\alpha + 2\beta + \sphericalangle AOB = 360° \Rightarrow 2\alpha + 2\beta = 360° - \sphericalangle AOB$.
Substituting $\sphericalangle AOB$ = 180° - $\sphericalangle BDA$, we have: $2\alpha + 2\beta = 180° + \sphericalangle BDA \Rightarrow \alpha + \beta = 90° + \sphericalangle BDA/2$.
Therefore $\angle BCA = 90° + \sphericalangle BDA/2 \Rightarrow \sphericalangle ACD = 90° - \sphericalangle BDA/2$
$\Rightarrow \sphericalangle CDA = 90° - \sphericalangle BDA/2 \Rightarrow DA = DC$.

Note that when $\sphericalangle AOB$ = 120°, ΔCDA will be an equilateral triangle (since $\sphericalangle BDA$ = 60° and $DA = DC$).

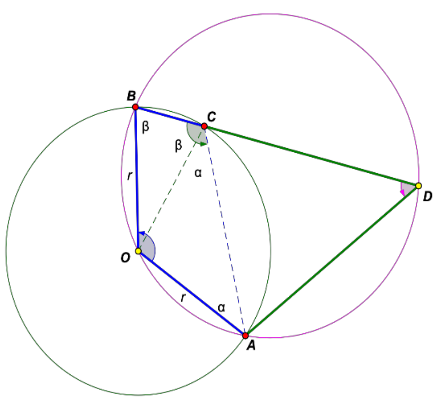

Figure 3

VARIATION

An interesting variation to explore is to consider what happens if a perpendicular from A is dropped on to BC extended to meet it at D as shown in Figure 4. In this case, we find that for a fixed $\sphericalangle AOB$, all right triangles CDA, as C is dragged along arc BA, are similar, since $\sphericalangle ACD$ and $\sphericalangle CDA$ are both constant. Additionally, since $\sphericalangle ACD$ = 180° - $\sphericalangle BCA$ = 180° - (180° - $\sphericalangle AOB/2$) = $\sphericalangle AOB/2$, it’s also easy to see that ΔCDA will be a 30°,60°,90° triangle when $\sphericalangle AOB$ = 60° or 120°.

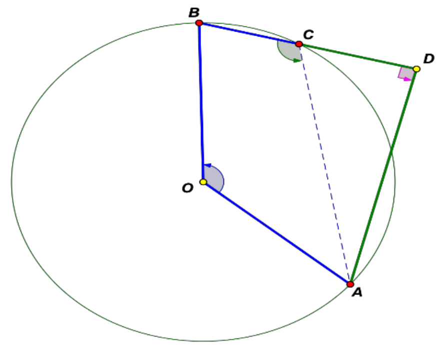

Figure 4

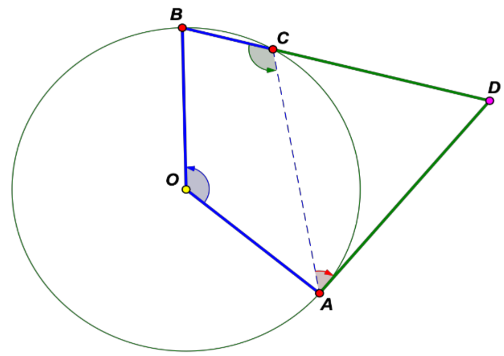

Figure 5

FURTHER VARIATION

Another interesting variation to explore is to consider what happens if instead $\sphericalangle CAD$ has a fixed value as shown in Figure 5. As in the preceding case, we also find that for a fixed $\sphericalangle AOB$, all triangles ΔCDA, as C is dragged along arc BA, are similar, since in this case $\sphericalangle ACD$ and $\sphericalangle CAD$ are both constant. Since $\sphericalangle ACD = \sphericalangle AOB/2$, it’s also easy to see that in this case ΔCDA will be an equilateral triangle when $\sphericalangle AOB$ = 120° and $\sphericalangle CAD$ = 60°.

More-over, the locus (path) of D is a circle through A, B and D as dynamically illustrated with an animation at the URL given earlier. This follows easily since for a fixed $\sphericalangle AOB$, $\sphericalangle CAD$ remains constant and is subtended by BA.

Lastly, note that the similarity of triangles CDA for a fixed $\sphericalangle AOB$, as well as the circular locus of D, also applies to the cases considered in Figures 2 and 3.

DYNAMIC INVESTIGATION

The introduction of dynamic geometry software (DGS) (in this case GeoGebra) into classrooms (high schools and colleges) creates a challenge to the praxis of theorem acquisition and deductive proof in the study and teaching of Euclidean geometry. Students/learners can experiment through different dragging modalities on geometrical objects that they construct, and consequently infer properties, generalities, and conjectures about the geometrical artifacts.

The dragging operation on a geometrical object enables students to apprehend a whole class of objects in which the conjectured attribute is invariant, and hence, the students become convinced that their conjecture will always be true Stupel et al, 2013). Nevertheless, because of the inductive nature of the DGE, we entitle this process 'semi proof'. Hence, following the employment of DGE, the experimental very small-theoretical gap that still exists in the acquisition and justification of geometrical knowledge becomes an important pedagogical and epistemological concern. Students must be aware that they still need to prove. Despite the fear, in our era, which is all accompanied by computerized technology, it is extremely important to integrate the technological tool in the teaching and research process.

CONCLUDING REMARKS

The above diagrams in conjunction with the use of dynamic geometry almost suffice as 'proofs without words' (compare Stupel et al, 2019). The given proofs of these results were therefore very short and cryptic, but learners and students should be expected to provide/include full reasons for every step in their own arguments.

This little investigation gives high school learners the opportunity to explain, prove and generalize an interesting result. It also very simply illustrates the value of drawing auxiliary lines as in drawing the radii in Figure 2, and how further reflection on a task, and considering variations, can lead to deeper geometric understanding.

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