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Another Student Discovery: The Quasi-Circumcentre and Quasi-Incentre of a Quadrilateral

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INTRODUCTION

A very gifted and inspirational South African mathematics teacher, Tickey de Jager (1921-2008), who taught at Rondebosch Boys' High School in Cape Town, many years ago advocated the naming of classroom discoveries after the students who proposed or discovered them. While such student discoveries are seldom original or novel, such a practice is nevertheless very encouraging and motivating for students. From personal experience I have certainly found over the years that students and learners at any level tend to work harder and much longer on problems they have discovered or formulated for themselves. It gives them a sense of personal ownership and nourishes their desire to solve such problems.

To enable such student or learner discoveries requires the regular use of open-ended classroom investigations during which they can be encouraged to ask their own 'what-if' questions (Brown & Walter, 1990). While engaging learners in creative problem solving through, for example, mathematics competitions like the SA Mathematics Olympiad (SAMO) is great, even greater benefits can be achieved by also helping students become problem posers. This can already be stimulated at primary school level, encouraging learners to critically reflect on a solved problem (in the style of Polya, 1945), considering what they have learnt from different ways of solving the problem to varying the conditions of the problem, or trying to generalize, specialize or apply it to other contexts.

To quote from Mason et al. (2010, pp. 139-140): "*From their earliest years, children can develop confidence to question, challenge and reflect. But they must be encouraged and reinforced in this. Their curiosity needs nurturing, their investigative potential structuring, their confidence maintaining. … If you are in a position to affect the learning of others, note how frequently you create the opportunity for* ***them*** *to think, to articulate their own questions, to challenge conjectures and to reflect on what has or has not been established.*"

To achieve this requires a mind-set change of the teacher, seeing him or herself as less of an authoritarian expert in the classroom and more of a collaborative facilitator. It means acknowledging that one sometimes doesn't know the answer straight away or beforehand, but to have the willingness to listen to and acknowledge pupils' questions and to work collaboratively on solving them. While time pressures, curriculum constraints and other factors often dampen spontaneous explorations that may arise in the classroom, learner exploration and discovery should be cherished and valued.

RENATE'S THEOREM ABOUT THE QUASI-CIRCUMCENTRE OF A QUADRILATERAL

Here is a personal example of one of the discoveries made by a student in a classroom discussion in 2006 at Kennesaw State University, while I was teaching on a visiting professorship there. Students were busy with the Water Supply task from De Villiers (1999) of finding the best position to build a water reservoir for four towns of more or less equal size. An interactive online sketch based on this activity is available at:
http://dynamicmathematicslearning.com/water-supply-four-towns.html

While the four given towns in the initial problem formed a cyclic quadrilateral, and hence had a unique equidistant point (the centre of the circle), the students had discovered from the activity (to their surprise) that not all quadrilaterals were cyclic. This raised the natural question: what would be the best position to build a water reservoir for four towns if they did not form a cyclic quadrilateral?

An undergraduate student, Renate Lebleu Davis, then proposed the intersection of the diagonals of the quadrilateral formed by the adjacent perpendicular bisectors of the (non-cyclic) quadrilateral as a possible solution (see Figure 2). While this was not the optimal solution I had in mind[1], I encouraged the class to further explore the properties of that point. Using dynamic geometry, the class very quickly came up with the following conjecture:

"Given a non-cyclic quadrilateral $ABCD$, let K, L, M and N be the respective circumcentres of triangles ABD, ABC, BCD and CDA, then the intersection O of KM and LN is equidistant from opposite vertices A and C, as well as equidistant from opposite vertices B and D. (Call this point O the *quasi-circumcentre*[2] of $ABCD$)".

Alternative formulation: let K, L, M and N be the respective intersections of the perpendicular bisectors of the adjacent sides of $ABCD$. For example, let K be the intersection of the perpendicular bisectors of sides AD and AB, etc.

A dynamic online sketch showing Renate's theorem is also available for readers at:
http://dynamicmathematicslearning.com/quasi-circumcentre-quad.html

But why was the result true? While the dynamic construction convinced them of the truth of the result no matter how they dragged the configuration, even into a concave as well as a crossed configuration, this empirical, experimental confirmation did not explain *why* the result was true.

Given the conceptual groundwork that had already been laid with the introductory activity about a perpendicular bisector of a line segment as the locus (path) of all points equidistant from the endpoints of the line segment (see Figure 1), it did not take long for the students, with some guidance, to come up with the following explanatory proof.

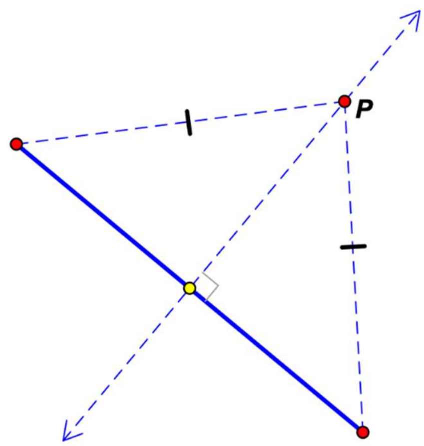

FIGURE 1: Perpendicular bisector as locus of equidistant points

[1] Mathematically, a more optimal solution can be obtained by minimizing the sum of the absolute values of all 6 differences between the four distances, which is equivalent to the least squares method.

[2] Initially we had chosen the name pseudo-circumcentre, but as it sometimes happens, later in the same year Myakishev (2006) proved the existence of a quasi-Euler line in relation to the same point, but calling it the quasi-circumcentre instead. This name seemed more appropriate, so we switched accordingly.

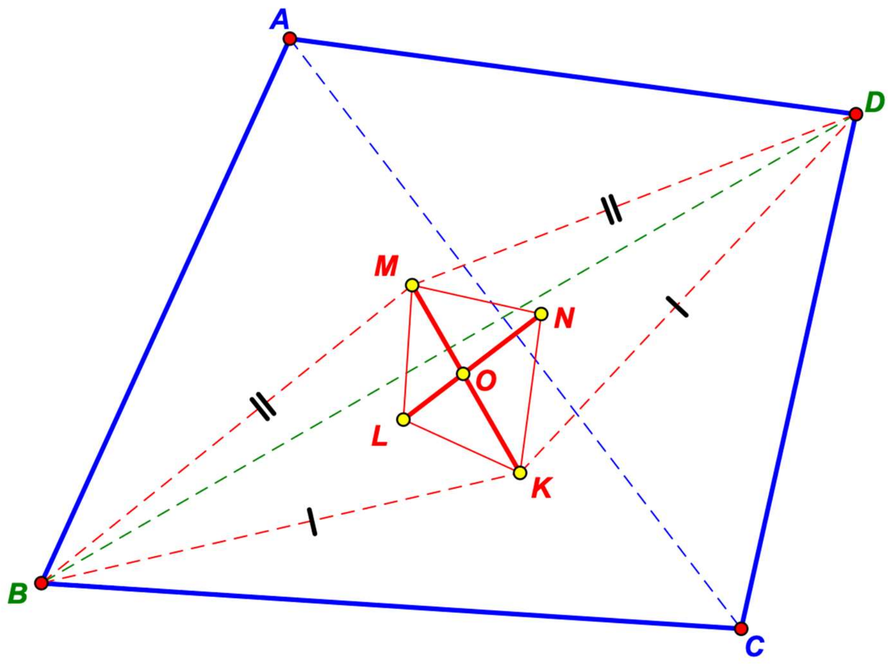

FIGURE 2: Quasi-circumcentre

Proof

Since both K and M lie on the perpendicular bisector of the diagonal BD, all points on the line KM are equidistant from B and D. Similarly, all points on the line LN are equidistant from A and C. Thus, the intersection O of lines KM and LN is equidistant from the two pairs of opposite vertices.

Renate's theorem was also later used in the Grade 11 Kennesaw State Mathematics Competition for High School students in 2007, as well as in the World InterCity Mathematics Competition for Junior High School students (up to Grade 9) in Durban in 2009. Noteworthy was that while the problem was experienced as one of most difficult ones for students participating in the Kennesaw competition, it was one of the easiest ones for the World Intercity competition with almost all primary and junior high school students from Asian countries getting full marks for it. None of the students from the South African team scored any marks for it. This clearly shows that – at least with respect to their mathematically talented learners – Asian countries appear to be engaging their students with much more in-depth, challenging geometry concepts and problems.

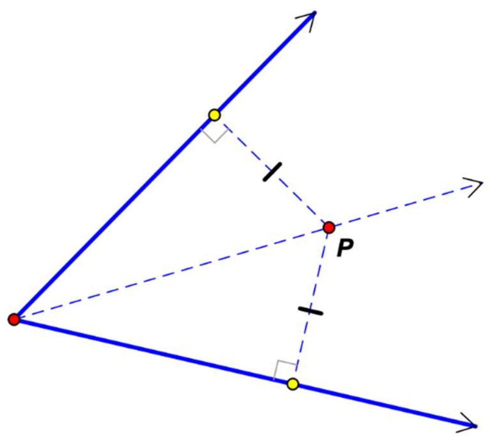

FIGURE 3: Angle bisector as locus of equidistant points

THE QUASI-INCENTRE OF A QUADRILATERAL

Since the angle bisector of an angle can, analogous to a perpendicular bisector, be seen as the locus of all points equidistant from the two rays forming the angle (see Figure 3), it was natural to next experimentally explore and formulate the following conjecture:

"Given a quadrilateral $ABCD$, construct the angle bisectors for each of the four angles as shown in Figure 4. Label E the intersection of the angle bisectors of angles A and B, label F the intersection of the angle bisectors of angles B and C, label G the intersection of the angle bisectors of angles C and D, and label H the intersection of the angle bisectors of angles D and A. Then the intersection I of EG and FH is equidistant from opposite sides AB and CD, as well as equidistant from opposite sides BC and DA. (Call this point I the *quasi-incentre* of $ABCD$)."

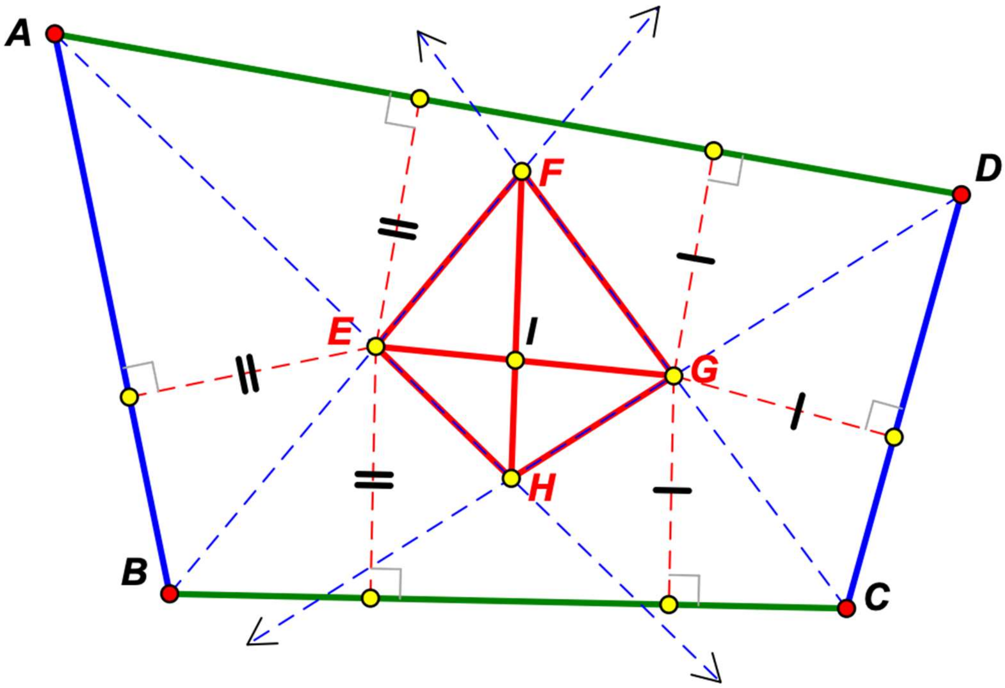

FIGURE 4: Quasi-incentre

Proof

Since E lies on both the angle bisectors of angles A and B, it is equidistant, by transitivity, from both AD and BC. Similarly, G is equidistant from AD and BC, and both H and F are equidistant from AB and CD. Hence, all points on the line EG are equidistant from AD and BC, and all points on the line FH are equidistant from AB and CD. Thus, the intersection I of EG and HF is equidistant from the two pairs of opposite sides.

A dynamic online sketch illustrating the quasi-incentre is also available at the same URL given earlier for the quasi-circumcentre.

CONCLUDING REMARKS

A recent study by Cai (2025, p. 164) reports that there has unfortunately been little progress in international curricula around the world to integrate problem posing into school mathematics, despite efforts by many over several decades. While it may sometimes feature in official policy statements, it has largely not penetrated the level of the intended curriculum – the prescribed curriculum, textbooks, and other learning materials – and even less at the level of the implemented (or attained) curriculum.

While Renate's theorem and its counterpart are not mathematically greatly significant, they appear to be fairly new and original. However, their value and significance for the students in the class was immeasurable – it gave them a sense of accomplishment and confidence in their own ability to discover and prove new mathematical results themselves. This little vignette also shows that it is possible for learners and students to be more active participants in classroom investigations.

It is therefore hoped that this little example will encourage other mathematics educators to create a fertile environment in their classes to provide similar opportunities for their learners and students to be creative and to ask and explore mathematical questions on their own or in collaboration with their teacher. Helping our learners and students become problem posers is as an important goal as developing their problem-solving skills. To quote Singer et al. (2013, p. 5): "*Problem posing is an old issue. What is new is the awareness that problem posing needs to pervade the education systems around the world, both as a means of instruction (…) and as an object of instruction* …"

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