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# Quasi-circumcenters and a Generalization of the Quasi-Euler Line to a Hexagon

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**Abstract**. This short note first proves an elementary property of the quasi-circumcenter of a quadrilateral, and then generalizes the quasi-Euler line of a quadrilateral to a hexagon involving its quasi-circumcenter, its quasi-orthocenter and its lamina centroid.

## 1. Introduction

The term "quasi-circumcenter" of a quadrilateral seems to have first been introduced by Myakishev in [2], where it is defined as follows: Given a quadrilateral $ABCD$, denote by $O_a$ the circumcenter of triangle $BCD$, and similarly, $O_b$, for triangle $ACD$, $O_c$ for triangle $ABD$, and $O_d$ for triangle $ABC$, then the quasi-circumcenter for the quadrilateral is given by $O = O_aO_c \cap O_bO_d$.

From a problem posed in [1] to find the "best" place to build a water reservoir for four villages of more or less equal size, if the four villages are not concyclic, the following theorem was experimentally discovered and proved. It followed from the classroom discussion of a proposed solution by an undergraduate student, Renate Lebleu Davis, at Kennesaw State University during 2006.

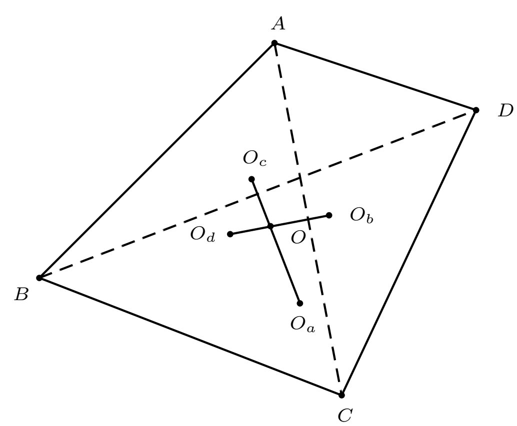


Figure 1

**Theorem 1.** *For a general quadrilateral $ABCD$, the quasi-circumcenter $O$ is equidistant from $A$ and $C$, and also from $B$ and $D$.* (See Figure 1).

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*Proof.* Since both $O_a$ and $O_c$ lie on the perpendicular bisector of the $BD$, all points on the line $O_aO_c$ are equidistant from $B$ and $D$. Similarly, all points on the line $O_bO_d$ are equidistant from $A$ and $C$. Thus, the intersection $O$ of lines $O_aO_c$ and $O_bO_d$ is equidistant from the two pairs of opposite vertices. □

This result was used in the Kennesaw State Mathematics Competition for High School students in 2007, as well as in the World InterCity Mathematics Competition for Junior High School students in 2009. Of interest too is that an analogous result exists as given below for the "quasi-incenter" of a quadrilateral, defined in the same way as quasi-circumcenter. The proof is left to the reader.

See note at bottom of page.

**Theorem 2.** *Given a general quadrilateral $ABCD$, then the quasi-incenter $I$ is equidistant from $AD$ and $BC$, as well as equidistant from $AB$ and $CD$.*

## 2. The quasi-circumcenter of a hexagon

The point of concurrency given in the Theorem 3 below defines the quasi-circumcenter of a hexagon.

**Theorem 3.** *If the quasi-circumcenters $P$, $Q$, $R$, $S$, $T$, and $U$, respectively of the quadrilaterals $ABCD$, $BCDE$, $CDEF$, $DEFA$, $EFAB$, and $FABC$ subdividing an arbitrary hexagon $ABCDEF$ are constructed, then the lines connecting opposite vertices of the hexagon formed by these quasi-circumcenters are concurrent.* (See Figure 2).

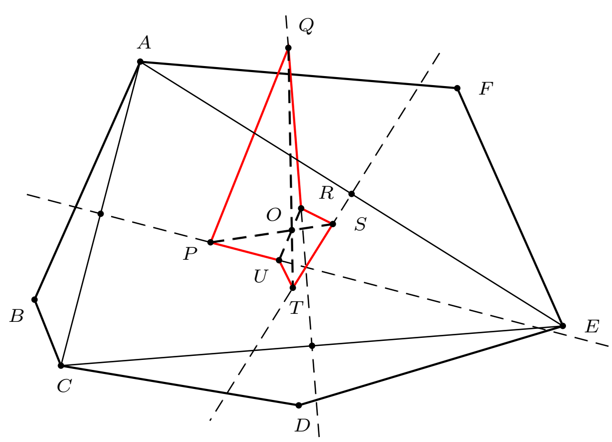


Figure 2

*Proof.* The result follows directly from the dual of the theorem of Pappus, which can be conveniently formulated as follows: *The diagonals of a plane hexagon whose sides pass alternatively through two fixed points, meet at a point.* With reference to Figure 2, note that alternate sides $QR$, $ST$ and $UP$ respectively lie on the perpendicular bisectors of sides $AE$, $EC$ and $CA$ of triangle $ACE$, and are therefore concurrent. Similarly, the other set of alternate sides are concurrent (with respect to triangle $BDF$). Hence, according to the dual of Pappus, the lines connecting the opposite vertices (the main diagonals) are concurrent. □

For clarity: the 'quasi-incentre' of a quadrilateral lies at the intersection of the diagonals of the quadrilateral formed by the angle bisectors of consecutive angles (and not by quadrilateral formed by the incentres of triangles ABC, BCD, CDA and DAB).

## 3. The quasi-Euler line of a hexagon

In [2], it is shown how the Euler line for a triangle generalizes to the Ganin - Rideau - Myakishev theorem, e.g. a quasi-Euler line for a general quadrilateral $ABCD$, which involves its lamina centroid $G$, its quasi-circumcenter $O$, and its quasi-orthocenter $H$ (which is defined in the same way as the quasi-circumcenter), and $OH : HG = 3 : -2$. Using the result of Theorem 3, this result generalizes to a hexagon as follows.

**Theorem 4.** *In any hexagon, its lamina centroid $G$, its quasi-circumcenter $O$, and its quasi-orthocenter $H$ are collinear, and $OH : HG = 3 : -2$.* (See Figure 3).

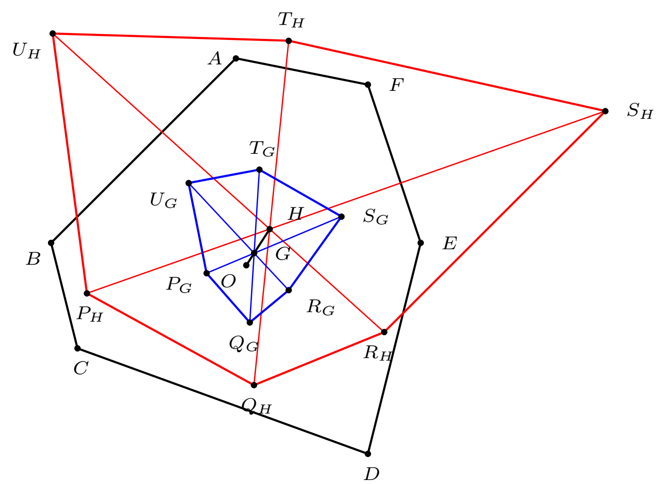


Figure 3

*Proof*. Subdivide the hexagon $ABCDEF$ into the same six quadrilaterals as in Theorem 3 above, and determine the quasi-circumcenter $O$, the lamina centroid $G$, and the quasi-orthocenter $H$ of each quadrilateral, respectively labelling the formed hexagons as

$$\mathcal{P}: PQRSTU, \qquad \mathcal{P}_G: P_GQ_GR_GS_GT_GU_G, \quad \mathcal{P}_H: P_HQ_HR_HS_HT_HU_H.$$

Since the same affine relations hold between $O$, $G$ and $H$ in each quadrilateral, affine mappings exist that will map $\mathcal{P}$ onto $\mathcal{P}_G$ and $\mathcal{P}_H$. But since the diagonals of $\mathcal{P}$ are concurrent, it follows that the diagonals of $\mathcal{P}_G$ and $\mathcal{P}_H$ would also be concurrent. Respectively label and define those two points of concurrency of $\mathcal{P}_G$ and $\mathcal{P}_H$ as the centroid $G$ and quasi-orthocenter $H$ for the whole hexagon. Finally, since affine transformations preserve collinearity as well as ratios into which segments are divided, we note from the affine mappings between the various hexagons that the result holds. □

We can similarly define the quasi-ninepoint center of a hexagon in terms of the six quasi-ninepoint centers of the subdividing quadrilaterals, from which it follows by the same affine transformations that the quasi-ninepoint center $N$ bisects the segment $OH$. An interactive Java applet to illustrate and explore Theorem 4 is available for the reader at:

`http://dynamicmathematicslearning.com/quasi-euler-line-hexagon.html`.

The result unfortunately does not generalize further to an octagon as subdividing it in the same way into quadrilaterals or hexagons do not produce octagons that generally have concurrent diagonals.

## References

[1] M. de Villiers, *Rethinking Proof with Sketchpad*, Emeryville: Key Curriculum Press, 1999/2003.

[2] A. Myakishev, On two remarkable lines related to a quadrilateral, *Forum Geom.*, 6 (2006) 289–295.

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