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Solutions to Reader Investigations: May 2004

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Unfortunately only one response was received to the *Reader Investigations* in the previous issue, and that was from Michael Fox, UK for Question 4. It is hoped that at least a few other readers or their learners worked on some of the problems below. However, an appeal is made to readers to please participate more actively in this section by making submissions or encouraging their students to do so.

1. Find all natural numbers x and y such that $x^2 - 4y = 3$.

Solution: There are no natural numbers x and y that satisfy this equation. There are two cases to consider:

(a) x is even. In this case there exists a natural number z such that $x = 2z$. Substitute this into the equation to obtain:

$$(2z)^2 - 4y = 4z^2 - 4y = 3$$

But the left side of this equation is divisible by 4 and the right side is not; therefore we have a contradiction.

(b) x is odd. In this case there exists a natural number z such that $x = 2z + 1$. Substitute this into the equation to obtain:

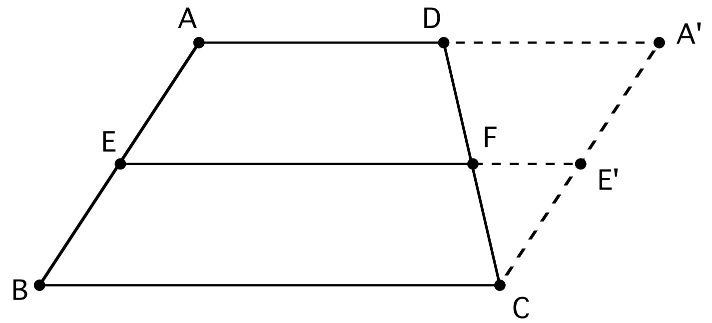
$$(2z+1)^2 - 4y = 4z^2 + 4z + 1 - 4y = 3$$

and subtracting 1 from both sides of this equation we have

$$4z^2 + 4z - 4y = 2$$

But as before the left side of this equation is divisible by 4 and the right side is not; therefore we have another contradiction.

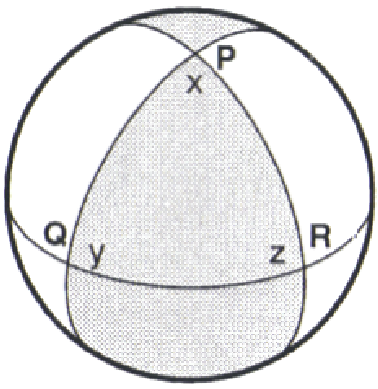
2. What relationship is there between the segment connecting the midpoints of the two non-parallel sides of a trapezium and the two parallel sides of the trapezium?



Solution: This segment EF is parallel to the two parallel sides of the trapezium and equal to half their sum (see sketch above). Probably the easiest (most straightforward) way to prove these is with co-ordinate geometry.

However a synthetic proof with a little transformation geometry is not difficult. Translate segment AB by vector BC. Then from the property of a translation and the 5th postulate of Euclid, line AA' is coincident with line AD. From translation, $EE' \,//\, BC \,//\, AD$. But $E'F \,//\, A'D$ since E' and F are both respective midpoints of CA' and CD. Therefore, again from 5th postulate, line $E'F$ coincides with line EE', and thus $EF \,//\, AD$. The remaining property now follows immediately from the figure by subtracting corresponding segments, since $2E'F = A'D$.

3. Investigate what the sum of the angles of a triangle is on a sphere and prove your observations.



Solution: A spherical triangle is formed by three great circles as shown above, but unlike a plane triangle its angle sum is always greater than 180° (or π radians). To understand why, we first need to find the area between two great circles separated by an angle of x radians. On the diagram above we consider the shaded area between the two great circles meeting at P.

The total angle at P is 2π radians and the area of the complete sphere is $4\pi R^2$. Hence the shaded area is

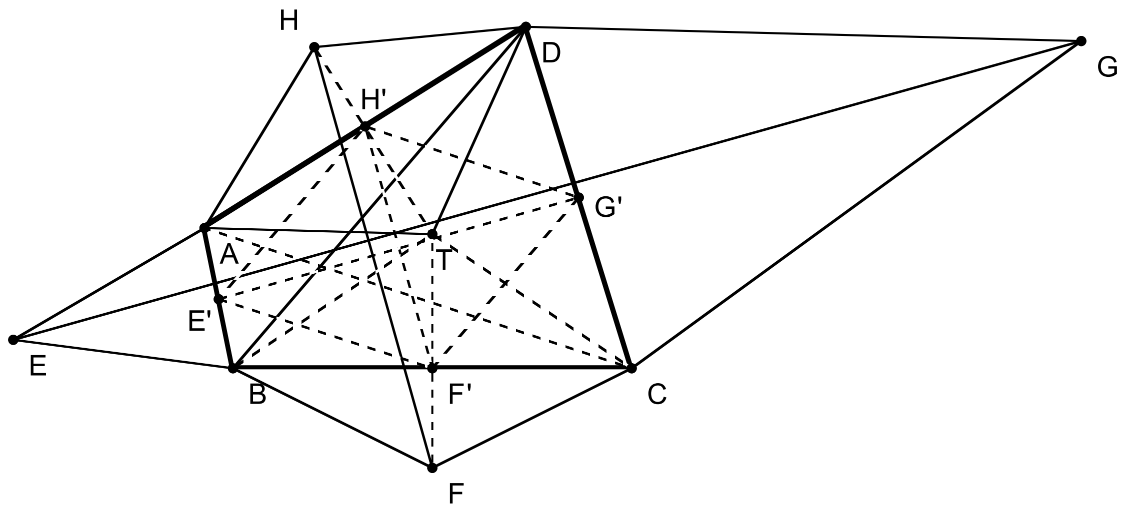
$$\frac{2x}{2\pi} \times 4\pi R^2 = 4R^2 x\,.$$

With this result we can now calculate the area A of the spherical triangle PQR. Taking the area between each pair of great circles in turn we include the area of the triangle six times. (Note: three times at the front as well as three more times at the back where the great circles intersect again). All the rest of the sphere is included only once, and these pieces together with the triangle PQR and its anti-pode at the back covers the whole sphere. Hence,

$$4R^2x + 4R^2y + 4R^2z = 4\pi R^2 + 4A$$
$$\therefore x + y + z = \pi + \frac{A}{R^2}$$

4. Given any quadrilateral $ABCD$ with $AC = BD$ and similar isosceles triangles constructed on the pairs of opposite sides as shown below, prove that $EG \perp FH$.

Two different, elegant proofs by Michael Fox from Leamington Spa, Warwickshire, UK, e-mail: mdfox@foxleam.freeserve.co.uk; are given below.



(a) Proof by congruence & similarity

With $AC = BD$, and triangles ADH, CBF isosceles and similar, as are BAE and DDG; we are to prove that EG and FH are perpendicular.

Let E', F', G', H' be the midpoints of AB, BC, CD, DA; and the perpendicular bisectors of DA and BC cut at T.

The proof is in two stages: first, that $E'G'$ and $F'H'$ are perpendicular; then that FH and EG are parallel to $F'H'$ and $E'G'$.

1. $E'F'G'H'$ is a rhombus, for $E'H' = F'G' = BD/2 = AC/2 = E'F' = G'H'$. (And, of course, $E'H' \parallel F'G'$, etc.) Thus $F'H'$ and $E'G'$, its diagonals, are perpendicular.

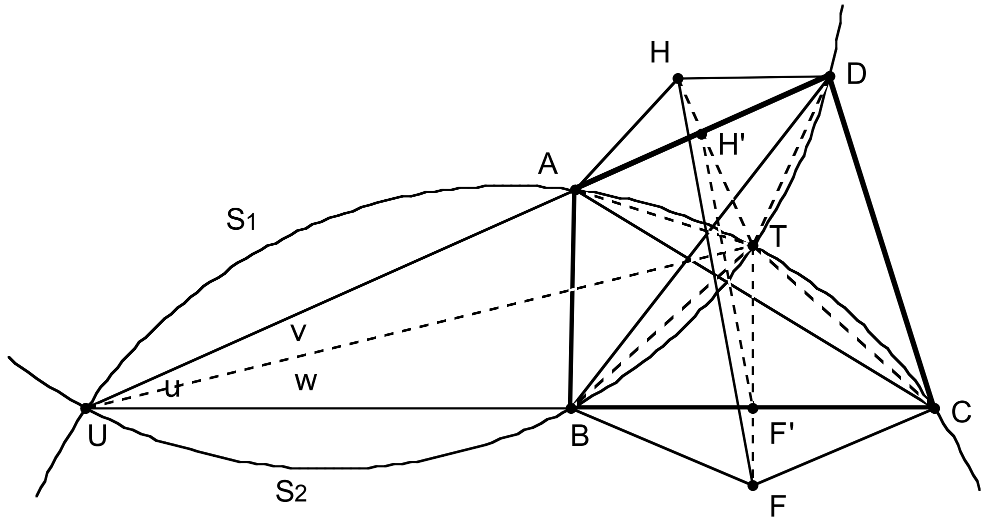
2. We prove next that triangles TDA and TBC are isosceles and similar.

Since T is on the perpendicular bisector of AD, $TD = TA$. Similarly $TB = TC$. Thus TDA and TBC are isosceles. We also have $BD = AC$. Hence triangles TDB, TAC are congruent. It follows that angles $DTA + ATB = DTB = ATC = ATB + BTC$, thus $DTA = BTC$, and the isosceles triangles TDB, TAC are similar.

We can now finish the proof.

From the similar triangles, $TH' : TF' = AD : BC$. But triangles ADH and CBF are similar, so $HH' : FF' = AD : BC$. Since T, H, H' are collinear, as are T, F, F', we have $TH' : TH = TF' : TF$, hence $FH \parallel F'H'$.

A similar argument shows that $EG \parallel E'G'$, and the result follows.



(b) *Proof with transformations*

Since ADH and CBF are similar triangles there is a spiral similarity that maps one on to the other (see above).

Let AD, BC meet at U. Draw circle S_1 through UAC, and S_2 through UBC.

Their second intersection, T, is the centre of the transformation.

Proof: Angles $BDT = BUT$ (in S_2) $= CUT$ (in S_1) $= CAT$; similarly $TBD = TUD = TUA = TCA$.

Thus triangles TDB, TAC are similar. Hence $TB : TD = TC : TA = k : 1$, say.

Also angles DTB, ATC are equal, each being opposite BUA in a cyclic quad, $= \theta$, say.

So a similarity (k, θ) takes TDA to TBC. Since ADH, CBF are similar, H will map on to F.

Thus quad $TDHA$ maps on to $TBFC$.

We must now establish that $TH'H$ is a line perpendicular to AD.

Proof: Chords AC of S_1 and BD of S_2 subtend the same angle at U, which is on both circles, therefore, since $AC = DB$, the circles are equal.

But chords AT and DT of these equal circles subtend the same angle v at U, and so are equal. Hence triangle TDA is isosceles, and $TDHA$ is a kite, so its diagonals are perpendicular.
Since $TF'F$ is the image of $TH'H$, we see that triangles $TH'F'$ and THF are similar, so $HF \parallel H'F'$.
The rest follows as in the first proof, though it would not be too hard to do a transformation proof of the final step.

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The Ten Commandments of Statistical Inference

I. Thou shalt not hunt statistical inference with a shotgun.

II. Thou shalt not enter the valley of the methods of inference without an experimental design.

III. Thou shalt not make statistical inference in the absence of a model.

IV. Thou shalt honour the assumptions of thy model.

V. Thou shalt not adulterate thy model to obtain significant results.

VI. Thou shalt not covet they colleagues' data.

VII. Thou shalt not bear false witness against thy control group.

VIII. Thou shalt not worship the 0.05 significance level.

IX. Thou shalt not apply large sample approximations in vain.

X. Thou shalt not infer causal relationships from statistical significance.

- Michael Driscoll (1977) in the *American Mathematical Monthly*, 84:628

(From Selkirk, K. & Willson, W.W. (1989). *Fifty Percent Proof*, Mathematical Association, p. 76.)