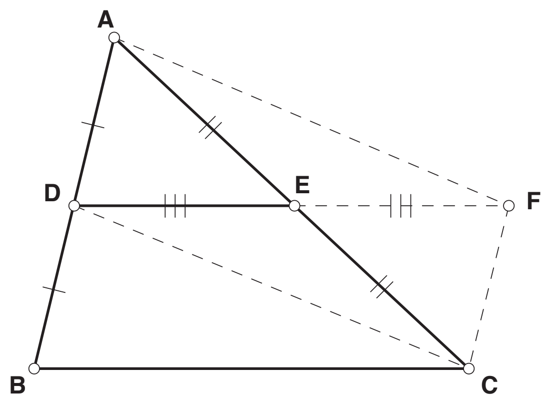
Reasoning Backward: Triangle Midpoints

In the activities Isosceles Trapezoid and Kite Midpoints, to explain (prove) our conjectures, we used the result that the midpoints of two sides of a triangle form a segment that is parallel to the third side and half its length. But why is this result true? Can we also prove it?



PROVING

Here are some hints for planning a possible proof. Read and work through them if you want or try to construct your own proof. If necessary, open the sketch **Triangle Midpoints.gsp** to help you answer the questions that follow.

Consider the figure above where it is given that $AD = DB$ and $AE = EC$. Extend $\overline{DE}$ to F so that $DE = EF$. Connect A and C with F, and D with C.

1. What can you conclude about quad *ADCF*? Why?

2. From Question 1, what can you conclude about $\overline{FC}$ and $\overline{AD}$?

3. From Question 2, what can you conclude about $\overline{FC}$ and $\overline{DB}$, and therefore about quad *DBCF*?

4. From Question 3, what can you conclude about $\overline{DF}$ and $\overline{BC}$, and therefore about $\overline{DE}$ and $\overline{BC}$?

Present Your Proof

Write out your explanation in a clear, systematic way, giving reasons for each step, and be ready to present it to the rest of the class.

Further Exploration

What is the converse of the theorem you just proved? Formulate it below and use Sketchpad to investigate whether it is true or not, producing a proof or a counterexample.

REASONING BACKWARD: TRIANGLE MIDPOINTS (PAGE 129)

This worksheet focuses on the systematization function of proof, since we are here constructing a proof for a result that was earlier discovered and accepted without proof. With the traditional deductive approach, this result and its proof would be presented before its application to results such as Varignon's theorem and kite and isosceles trapezoid midpoints. However, in actual mathematical research, results are seldom discovered in this straightforward linear fashion. For example, we might first discover an interesting result (for example, Varignon's theorem) and then, upon trying to prove it, find that it can be proved in terms of another result (triangle midpoints). Our attention then shifts to proving this other result (the lemma, if you like). In writing up the results and their proofs, it is of course conventional to first prove the lemma and then the main result, but if this is used as a *teaching approach,* it hides the fact that the actual sequence of discovery may have been the other way around. This worksheet attempts to give students some insight into the way a deductive ordering of some results may be arrived at by reasoning backward, rather than pretending that we always have the phenomenal foresight to first prove a particular, relatively uninteresting theorem (or lemma) because we anticipate that it will be used in proving important, interesting results later on.

Prerequisites: Kite Midpoints, Isosceles Trapezoid, and Logical Discovery (Varignon) activities in this book, and knowledge of the properties of and conditions for a parallelogram.

Sketch: Triangle Midpoints.gsp.

PROVING

1. $ADCF$ is a parallelogram because its diagonals $\overline{AC}$ and $\overline{DF}$ bisect each other.
2. $\overline{FC} \parallel \overline{AD}$ because opposite sides of a parallelogram are equal and parallel.
3. $\overline{FC} \parallel \overline{DB}$ because $AD = DB$ and ADB is a straight line. Therefore, $DBCF$ is a parallelogram (opposite sides are equal and parallel).
4. $\overline{DF} \parallel \overline{BC}$ because they are opposite sides of parallelogram $DBCF$. Therefore, $DE = \frac{1}{2}BC$ and $\overline{DE} \parallel \overline{BC}$.

Further Exploration

If from the midpoint of a side of a triangle a line is drawn parallel to another side, this line bisects the third side.

Although the proof is similar to the preceding one, some students may need your help. Draw $\overline{CF} \parallel \overline{BD}$ and extend $\overline{DE}$ to meet $\overline{CF}$ to form the parallelogram $DBCF$. The rest of the proof is then similar to the preceding proof, but in reverse.