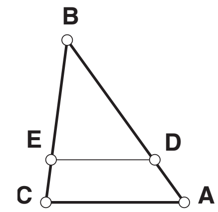
Name(s):

Parallel Lines

CONJECTURE



- Open the sketch **Parallel.gsp.** Drag different points in your sketch. Notice that point D is a free point on $\overline{AB}$ of ΔABC.
- Press the button that draws a segment from point D.

1. Drag point D and then complete this statement: $\overline{ED}$ ________________ $\overline{CA}$.

- Press the button that draws a segment from point E.

2. Drag point D again and then complete this statement: $\overline{EF}$ ________________ $\overline{BA}$.

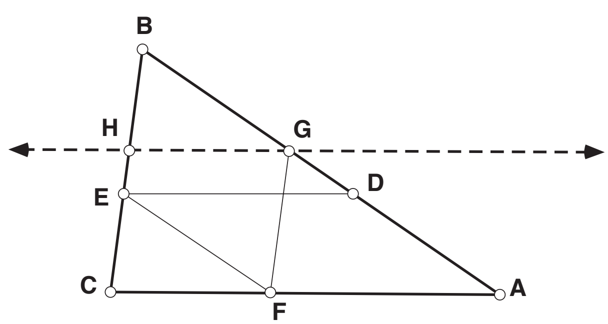
- Press the button that draws a segment from point F.

3. Drag point D again and then complete this statement: $\overline{FG}$ ________________ $\overline{BC}$.

4. Make sure that points D and G are not overlapping. Do you think you would ever come back to your starting point D if you continued drawing parallel segments to the sides?

If you don’t think you would come back to your starting point D, why not? If you think you would, under what conditions, and after drawing how many parallel segments?

- Construct at least another three parallel lines continuing the pattern of the first three segments.



5. What do you notice? Drag point D and any of the vertices of ΔABC to check your observation.

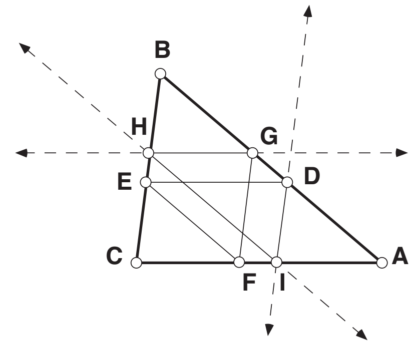
To construct your first parallel line, select point G, then $\overline{AC}$, and choose **Parallel Line** from the Construct menu.

6. How certain are you that your conjecture is always true? Record your level of certainty on the number line and explain your choice.

CHALLENGE If you believe your conjecture is always true, provide some examples to support your view and try to convince your partner or members of your group. Even better, support your conjecture with a logical explanation or a convincing proof. If you suspect your conjecture is not always true, try to supply counterexamples.

PROVING

You should have noticed that if you construct parallel lines as described, you need to go around only twice (constructing a total of six parallel lines) before you return to your starting point *D*. (When *D* is at the midpoint of *AB*, you need to go around only once, constructing three parallel lines.) Most people find this surprising, thinking instead that in some instances you might never return to the beginning. How can we convince ourselves that this is *always* the case?

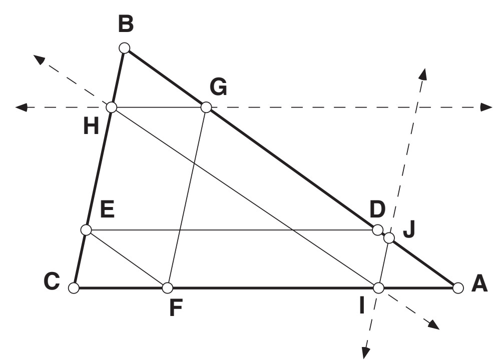


Press the button to show the ratios $\frac{BD}{DA}$ and $\frac{BE}{EC}$.

7. Drag point *D* and any vertex of ΔABC to look for patterns. What do you notice about these ratios? This will show a pattern in a triangle that you may already have proved or discovered.

You will use your results from Question 7 in the rest of your proof.

To continue with a proof of your original conjecture from Question 5, you can use a form of proof called *proof by contradiction.* To use proof by contradiction, start by assuming that your conclusion is false. Then show that this leads to a contradiction. In this activity, assume that you do not return to point *D* after constructing six parallels. Assume instead that you return to some different point *J*, as in the picture. Otherwise, the picture matches your construction so far: $\overline{DE} \parallel \overline{GH} \parallel \overline{AC}$, $\overline{FG} \parallel \overline{IJ} \parallel \overline{CB}$, and $\overline{FE} \parallel \overline{IH} \parallel \overline{AB}$. Can you now logically show that *J must* coincide with *D*?



First try it on your own, but if you get stuck, read and work through the following for planning a possible proof.

8. Use your result from Question 7 to continue the sequence of equations relating all the ratios into which the sides are divided by the points *D*, *E*, *F*, *G*, *H*, *I*, and *J*.

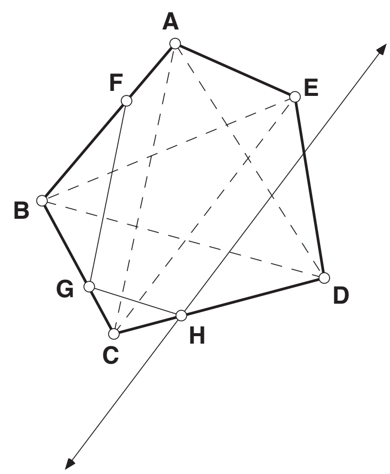
$$\frac{BD}{DA} = \frac{BE}{EC} = \frac{AF}{FC} = \underline{\hspace{4cm}}$$

9. What do your equations say about $\frac{AD}{DB}$ and $\frac{AJ}{JB}$? What can you conclude from this?

Presenting Your Proof

Look over Questions 7–9. Now write a proof of your conjecture in your own words. You may include a demonstration sketch to support and explain your proof.

Further Exploration



1. What happens if one or more of the points *D* through *I* fall on the extensions of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$? Does your result still hold?

2. What happens if in a pentagon *ABCDE*, a segment $\overline{FG}$ is drawn parallel to $\overline{AC}$ from a point *F* on $\overline{AB}$, a segment *GH* is drawn parallel to $\overline{BD}$, and so on? Would we ever come back to point *F*? Prove your observations.

3. Generalize your observation in Question 2 to polygons with a similar property.

PARALLEL LINES (PAGE 95)

This worksheet also focuses on emphasizing the verification function of proof, since most people tend to find it rather surprising that the parallel lines will always return to the original starting point. Intuitively, most people guess that it would depend on the position of *D*, and that in some cases we can carry on parallel lines indefinitely without their returning to their starting point. The discovery, therefore, seems a little counterintuitive, which makes it a good context for emphasizing the verification function of proof.

Prerequisite: None.

Sketch: Parallel.gsp.

CONJECTURE

1. $\overline{ED} \parallel \overline{CA}$.
2. $\overline{EF} \parallel \overline{BA}$.
3. $\overline{FG} \parallel \overline{BC}$.
4. Responses will vary.
5. We need only go around twice (i.e., draw six parallel lines) before we return to *D*.
6. Responses may vary.

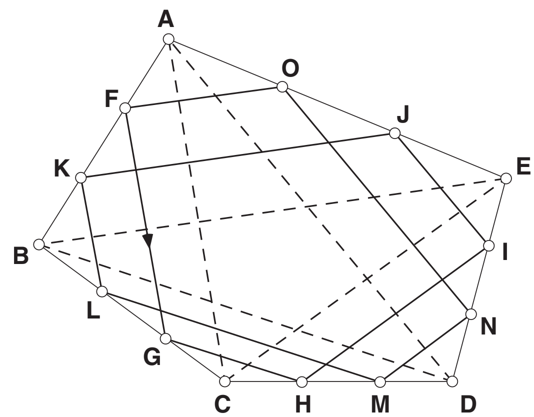
CHALLENGE It is important for you, as the teacher, to take a neutral stand here, or even better that of a skeptic, and not to indicate to the students that the result is indeed true. Challenge them to convince you or other skeptics in the class.

PROVING

7. The parallel lines divide adjacent sides into equal ratios. (This result will be proved later on, in Chapter 5.)
8. $\frac{BD}{DA} = \frac{BE}{EC} = \frac{AF}{FC} = \frac{AG}{GB} = \frac{CH}{HB} = \frac{CI}{IA} = \frac{BJ}{JA}$
9. $\frac{BD}{DA} = \frac{BJ}{JA}$; therefore, *J* must coincide with *D*.

Further Exploration

1. The result still holds, even if point *D* lies on the extension of the side *AB,* in which case the other points will also lie on the extensions of the other sides.
2. In a pentagon, we also need only go around twice (draw 10 lines) to return to our starting point (see below). The proof is similar to the preceding one.



3. The result is generalizable to any polygon with an odd number of vertices. A precise formulation and a general proof is given in de Villiers (1996, 83–85), as well as a similar result for polygons with an even number of vertices.

Related Results

You might also ask your students to measure the areas and perimeters of the hexagon and compare it with the original triangle. They will then discover that the ratio of the area of the hexagon *DEFGHI* to that of triangle *ABC* is also constant for a fixed position of the starting point *D* (see figures on the following page). A proof of this result is given in de Villiers (1999b), but requires a definition of the area of crossed polygons, which would probably be beyond most high school students. (See the discussion regarding the area of a crossed quadrilateral in the Teacher Notes for the Varignon Area activity.)

On the other hand, the perimeter of the hexagon *DEFGHI* is always equal to that of triangle *ABC,* irrespective of the position of *D;* this is very easy to prove.