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REVISITING THE VAN HIELE THEORY

YouTube Video Recording: https://www.youtube.com/watch?v=qBNiYGbfyII

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Abstract: This paper gives a review of research on the Van Hiele Theory over the past 30 years, and highlights some important issues regarding theoretical implications for specifically designing learning activities in dynamic geometry contexts, as well as issues for further research such as the role of proof and hierarchical class inclusion.

Keywords: Van Hiele theory, dynamic geometry, levels of thinking, conceptual structuring.

Introduction

The Van Hiele theory originated in the respective doctoral dissertations of Dina van Hiele-Geldof and her husband Pierre van Hiele at the University of Utrecht, Netherlands in 1957. While Pierre's dissertation mainly tried to explain why pupils experienced problems in geometry education (in this respect it was **explanatory** and **descriptive**), Dina's dissertation was about a teaching experiment and in that sense is more **prescriptive** regarding the ordering of geometry content and learning activities of pupils. The most obvious characteristic of the theory is the distinction of five discrete thought levels in respect to the development of pupils' understanding of geometry.

The main reason for the failure of the traditional geometry curriculum was attributed by the Van Hieles to the fact that the curriculum was presented at a higher level than those of the pupils; in other words they could not understand the teacher nor could the teacher understand why they could not understand! Although the Van Hiele theory distinguishes between five different levels of thought, we shall here only focus on the first four levels as they are the most pertinent ones for secondary school geometry. The general characteristics of each level can be described as **Visual Recognition**, **Analysis of Properties**, **Ordering**, and **Deduction**.

Note that in a certain sense the transition from Level 1 to Level 2 involves a transition from an inactive-iconic handling of concepts to a more symbolic one, to use Bruner's familiar concepts. More simply put, the attainment of Level 2 involves the acquisition of the technical language by which the properties of the concept can be described. However, the transition from Level 1 to Level 2 involves more than just the acquisition of language. It involves recognizing certain new relationships between concepts and the refinement and renewal of existing concepts.

For a student to progress from Level 1 to Level 2 regarding a particular topic (e.g. the quadrilaterals), a significant re-arrangement of relationships and a refinement of concepts have to occur. There is therefore far more in this transition than merely a verbalization of intuitive knowledge; the verbalization goes together with a restructuring of knowledge. This restructuring must first occur before students can start exploring the logical relationships between these properties at Level 3.

Level 3 also represents a completely different network of relations than Level 2. Where the network of relations at the Level 2 involves the *association of properties* with types of figures and relationships between figures according to these properties, the network of relations at the Level 3 involve the *logical relationships* between the properties of figures. The network of relations at the Level 3 no longer refer to concrete, specific figures, nor do they form a frame of reference in which it is asked whether a given figure has certain properties. The typical questions that are asked at Level 3 are whether a certain property follows from another, or can be deduced from a particular subset of properties (in other words whether it could be taken as a definition or is a theorem) or whether two definitions are equivalent.

The network of relations for the First and Second thought levels are therefore quite different from the Third (Van Hiele, 1973, p. 90):

> The reasoning of a logical system belongs to the Third Level of thought. The network of relations, which is based on a verbal description of observed facts, belongs to the Second Level of thought. These two levels have their own networks of relations where the one is distinct from the other: one either reasons in the one network of relations or in the other.

The primary and middle school geometry curriculum

In South Africa we still have a geometry curriculum heavily loaded in the senior secondary school with formal geometry, and with relatively little content done informally in the primary school. (E.g. little similarity or circle geometry is done in the primary school). On average, pupils' performance in the South African matric (Grade 12) geometry was far worse than in algebra. Why?

The Van Hiele Theory supplies an important explanation. For example, research by De Villiers and Njisane (1987) showed that about 45% of pupils investigated in Grade 12 in KwaZulu had only mastered Level 2 or lower, whereas the examination assumed mastery at Level 3 and beyond! Similar low Van Hiele levels among secondary school pupils have been found by Malan (1986), Smith and De Villiers (1990) and Govender (1995). In particular, the transition from Level 1 to Level 2 posed specific problems to second language learners, since it involves the acquisition of the technical terminology by which the properties of figures need to be described and explored. This requires sufficient time, which is not available in the presently overloaded secondary curriculum.

It seems clear that no amount of effort and fancy teaching methods at the secondary school will be successful, unless we embark on a major revision of the primary school geometry curriculum along Van Hiele lines. The future of secondary school geometry thus ultimately depends on primary school geometry!

In Japan for example pupils already start off in Grade 1 with extended tangram, as well as other planar and spatial, investigations (e.g. see Nohda, 1992). This is followed up continuously in following years so that by Grade 5 they are already dealing formally with the concepts of congruence and similarity; concepts that are only introduced in Grades 8 and 9 in South Africa. Similarly in Taiwan where geometry is started early, it is reported in a study by Wu and Ma (2006) that 28.3% of Grade 6 learners were already at Van Hiele 3, whereas the same percentage of learners at Van Hiele Level 3 in South

Africa, only occurred in Grade 11 (De Villiers & Njisane, 1987; De Villiers, 1987). More recently, Feza and Webb (2005) found that only 5 out of 30 (16.7%) Grade 7 learners interviewed in South Africa, had reached Van Hiele Level 2. It seems no wonder that in international comparative studies in recent years, Japanese and Taiwanese school children have consistently outperformed school children from South Africa, as well as other countries.

Although the recent introduction of tessellations in South African primary schools is to be greatly welcomed, many teachers and textbook authors do not appear to understand its relevance in relation to the Van Hiele theory. Although tessellations have great aesthetic attraction due to their intriguing and artistically pleasing patterns, the fundamental reason for introducing it in the primary school is that it provides an intuitive visual foundation (Van Hiele 1) for a variety of geometric content, which can later be treated more formally in a deductive context.

For example, in a triangular tessellation pattern such as shown in Figure 1, one could ask pupils the following questions:

(1) identify and colour in parallel lines

(2) what can you say about angles *A*, *B*, *C* , *D* and *E* and why?

(3) what can you say about angles *A*, *1*, *2*, *3* and *4* and why?

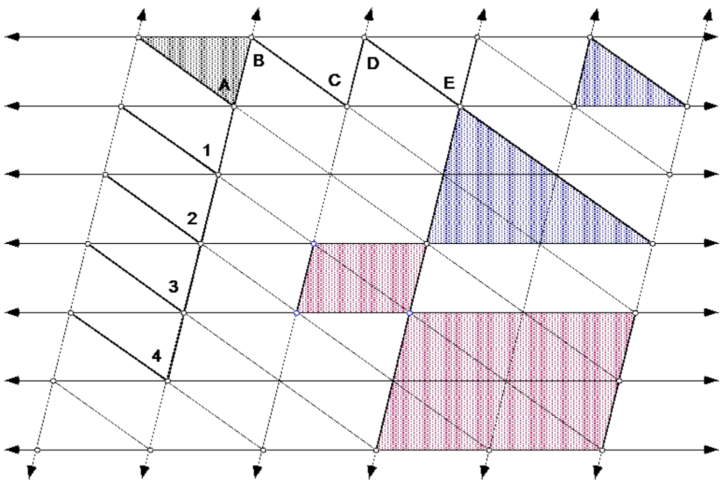

Figure 1. Visualization

In such an activity pupils will realize that angles *A*, *B*, *C*, *D* and *E* are all equal since a half turn of the grey triangle around the midpoint of the side *AB* maps angle *A* onto angle *B*, etc. In this way, pupils can be introduced for the first time to the concept of "*saws*" or "*zig-zags*" (alternate angles). Similarly, pupils should realize that angles *A*, *1*, *2*, *3* and *4* are all equal since a translation of the grey triangle in the direction of angles *1*, *2*, *3* and *4* consecutively maps angle *A* onto each of these angles. In this way, pupils can be introduced for the first time to the concept of "*ladders*" (corresponding angles). Pupils should further be encouraged to find different saws and ladders in the same and other tessellation patterns to improve their visualization ability.

Since each tile has to be identical and can be made to fit onto each other exactly by means of translations, rotations or reflections pupils can easily be introduced to the concept of congruency. Pupils can also be asked to look for different shapes in such tessellation patterns, e.g. parallelograms, trapezia and hexagons. They could also be encouraged to look for larger figures with the *same shape*, thus intuitively introducing them to the concept of *similarity* (as shown in Figure 1 by the shaded similar triangles and parallelograms).

Tessellations also provide a suitable context for the analysis of the properties of geometric figures (Van Hiele 2), as well as their logical explanation (Van Hiele 3). For example, after pupils have constructed a triangular tessellation pattern as shown in Figure 2, one could ask them questions like the following:

(1) What can you say about angles *A* and *B* in relation to *D* and *E*? Why? What can you therefore conclude from this?

(2) What can you say about angles *F* and *G* in relation to angles *H* and *I*? Why? What can you therefore conclude from this?

(3) What can you say about line segment *JK* in relation to line segment *LM*? Why? What can you therefore conclude from this?

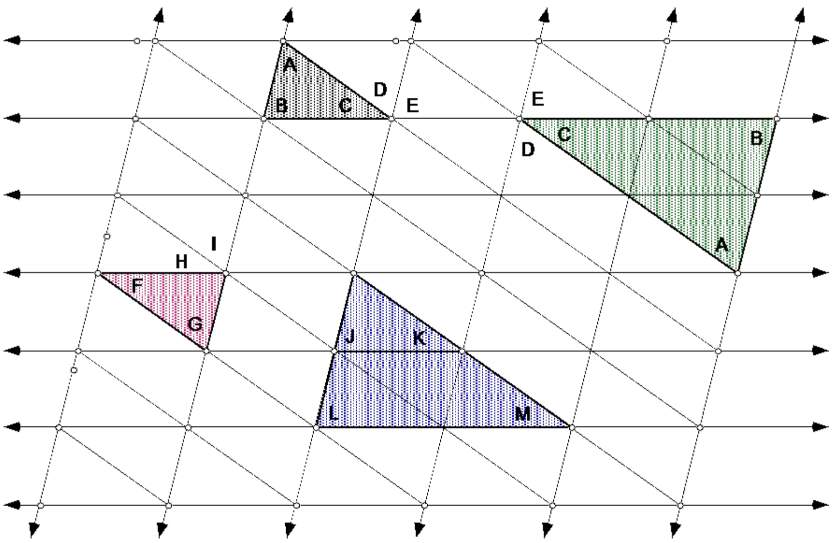

Figure 2. Analyzing

In the first case, pupils can again see that angle *A* = angle *D* due to a saw being formed. Also angle *B* = angle *E* due to a ladder. It is then easy for them to observe that since the three angles lie on a straight line, that the sum of the angles of triangle *ABC* must be equal to a straight line. They can also observe that this is true at any vertex, as well as for any size triangle or orientation, thus enabling generalization. In the second case, the exterior angle theorem is introduced and in the third case, the midpoint theorem. Such analyses are clearly just a short step away from the standard geometric explanations (proofs); all they now need is some formalization. In Figure 3 the three levels are illustrated for the discovery and explanation that the opposite angles of a parallelogram are equal.

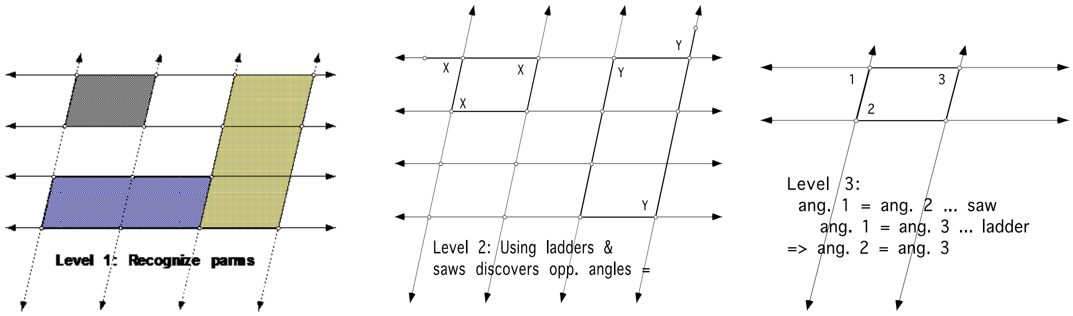

Figure 3. Three levels

Conceptual structuring

A very important aspect of the Van Hiele theory is that it emphasizes that informal activities at Levels 1 and 2 should provide appropriate "*conceptual substructures*" for the formal activities at the next level. Though different, this idea is somewhat similar to the idea of instructional *scaffolding* promoted by Wood, Bruner and Ross (1976).

Teachers often let their students measure the angles of a triangle with a protractor, and then let them add the angles (usually disregarding 'deviations' as due to experimental error) to 'discover' that they always add up to 180°.

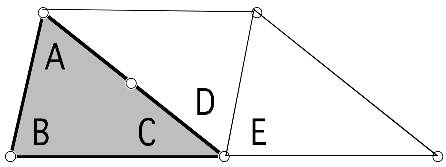

Figure 4. Using transformations to discover

However, from a Van Hiele perspective this is entirely inappropriate as it does not provide a suitable conceptual substructure in which the eventual logical explanation (proof) is implicitly embedded. In comparison, an activity with cardboard tiles or *Sketchpad* like the following from De Villiers (2003) provides such a substructure. For example, translate a triangle *ABC* by vector *BC*, and rotate triangle ABC around the midpoint of *AC* (see Figure 4). Let the students notice through dragging that the three angles at *C*, *D*, and *E* always form a straight line. Then ask students what they can say about angles *A* and *B* in relation to angles *D* and *E* in terms of the transformations carried out. Since angle *B* maps on to angle *E* by the translation, and angle *A* maps to angle *D* by the half-turn, angles *B* and *A* are equal to angles *D* and *E*, respectively. Clearly this provides a much more appropriate conceptual structure for an eventual explanation (proof) than simply letting students measure some angles of triangles.

Similarly, the activity of measuring the base angles of an isosceles triangle is conceptually inappropriate, but folding it around its axis of symmetry lays the foundation for a formal proof later. The same applies to the investigation of the properties of the quadrilaterals. For example, it is conceptually inappropriate to measure the opposite angles of a parallelogram to let pupils discover that they are equal. It is far better to let them give the parallelogram a half-turn to find that opposite angles (and sides) map onto each other, as this generally applies to all parallelograms and contains the conceptual seeds for a formal proof.

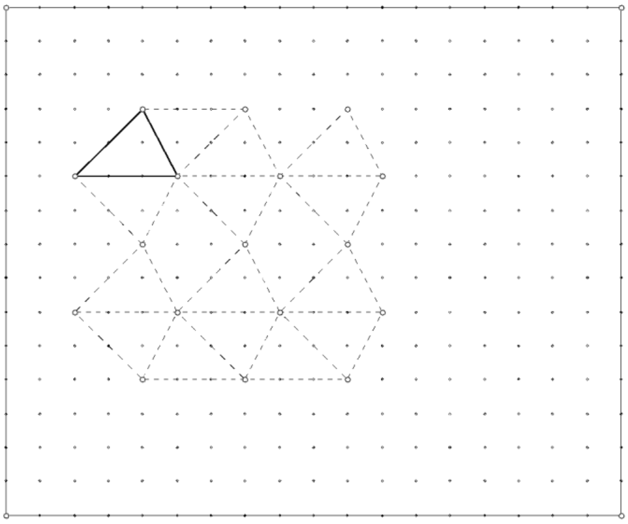

Figure 5. Using grids to produce tessellations

Recently I had a conversation with a teacher who quickly dismissed a fellow teacher's introduction to tessellations who first let his pupils pack out little cardboard tiles. This teacher felt that it produced untidy patterns, was ineffective and time consuming, and that one should just start by providing pupils with ready-made square or triangular grids and show them how they can then easily draw neat tessellation patterns (see Figure 5). Although such grids are a useful and effective way of drawing neat patterns, it is conceptually extremely important for pupils to at least have some experience of physically packing out tiles, i.e. rotating, translating, reflecting the tiles *by hand* (or at the very least with the aid of dynamic geometry software, illustrating or animating the underlying transformations).

The first problem is that it is possible to draw tessellation patterns on such grids without any clear understanding of the underlying isometries by which they can be created, which in turn are conceptually important for analyzing the geometric properties embeded in the pattern, and eventually for formalizing them into proofs.

More importantly, according to Bruner this *enactive* level, where the child manipulates materials like tiles directly, is a fundamental **prerequisite** (just as in the Van Hiele theory), for the *iconic* level, where the child now begins to deal with mental images of objects and no longer needs to manipulate them directly.

Defining and classifying

Traditionally most teachers and textbook authors have simply provided students with ready-made content (definitions, theorems, proofs, classifications, and so on) that they merely have to assimilate and regurgitate in tests and exams. Traditional geometry education of this kind can be compared to a cooking and bakery class where the teacher only shows students cakes (or even worse, only pictures of cakes) without showing them what goes into the cake and how it is made. In addition, they're not even allowed to try their own hand at baking!

Mathematicians and mathematics educators alike have often criticized the direct teaching of geometry definitions with no emphasis on the underlying process of defining. The well-known mathematician Hans Freudenthal (1973, pp. 416-418) also strongly criticized the traditional practice of the direct provision of geometry definitions as follows:

> ... the Socratic didactician would refuse to introduce the geometrical objects by definitions, but wherever the didactic inversion prevails, deductivity starts with definitions.
>
> ... most definitions are not preconceived but the finishing touch of the organizing activity. The child should not be deprived of this privilege ... Good geometry instruction can mean much - learning to organize a subject matter and learning what is organizing, learning to conceptualize and what is conceptualizing, learning to define and what is a definition. It means leading pupils to understand why some organization, some concept, some definition is better than another.

Just knowing the definition of a concept does not at all guarantee understanding of the concept. For example, although a student may have been taught, and be able to recite, the standard definition of a parallelogram as a quadrilateral with opposite sides parallel, the student may still not consider rectangles, squares and rhombi as parallelograms, since the students' concept image of a parallelogram is that not all angles or sides are allowed to be equal.

According to the Van Hiele theory understanding of formal, textbook definitions only develops at Level 3, and that the direct provision of such definitions to students at lower levels would be doomed to failure. In addition, if we take the constructivist theory of learning seriously (namely that knowledge simply cannot be transferred directly from one person to another, and that meaningful knowledge needs to be actively (re)-constructed by the learner), students ought be engaged in the activity of defining and allowed to choose their own definitions at each level. This implies allowing the following possible kinds of meaningful definitions for a rectangle at each Van Hiele level:

Van Hiele 1

Visual definitions; for example, a rectangle is a figure which looks like this (draws or identifies a quadrilateral with all angles 90° and two long and two short sides).

Van Hiele 2

Uneconomical definitions; for example, a rectangle is a quadrilateral with opposite sides parallel and equal, all angles 90°, equal diagonals, half-turn-symmetry, two axes of symmetry through opposite sides, two long and two short sides, etc.

Van Hiele 3

Correct, economical definitions; for example, a rectangle is a quadrilateral with two axes of symmetry though opposite sides.

Hierarchical *versus* Partition Definitions

Though children at an early age are capable of understanding class inclusions like "cats and dogs are animals", it appears substantially more difficult to accomplish with geometric figures. Generally, students' spontaneous definitions at Van Hiele Levels 1 and 2 as shown above would also tend to be *partitional*; in other words, they would not allow the inclusion of the squares among the rectangles (by explicitly stating two long and two short sides). In contrast, according to the Van Hiele theory, definitions at Level 3 are typically *hierarchical*, which means they allow for the inclusion of the squares among the rectangles, and would not be understood by students at lower levels.

In traditional instruction children are mostly introduced to rectangles, rhombi, parallelograms, etc. as '*static geometric objects*'. For example, a rectangle might be introduced by comparison to the shape of a door or a static picture in a book, but a door or a picture in a book cannot be transformed into a square (unless parts are cut off). So the concept rectangle is from the start introduced as a concept completely disjoint from a square. Unfortunately this partition classification schema then becomes entrenched and fossilized over time, and appears very resistant to change.

The conceptual difficulty of geometric class inclusion was already shown by Mayberry (1981) who found that only 3 out of 19 preservice mathematics teachers indicated squares also as rectangles on a sheet of some given quadrilaterals. Though valid criticism can be raised against some of questions used by Mayberry, as well as by Usiskin (1982) to evaluate hierarchical thinking, since given a set of different quadrilaterals, students might just mark the most general quadrilateral (e.g. a general parallelogram) when asked to mark it, simply *not knowing or realizing the intention* of the question was that all the special cases (e.g. rectangles, rhombi & squares) had to be marked as well.

In research conducted by De Villiers and Njisane (1987) with 4015 students from KwaZulu (South Africa) with some modified questions for evaluating hierarchical thinking (see Figure 6 for an example), some small improvement was observed. Nonetheless, it was found that very little progress occurred in their hierarchical thinking from Grade 9 to Grade 12, only ranging from 0.5% to 5.1% success with a 50% criterion on test items evaluating hierarchical thinking. This contrasts starkly with Van Hiele 3 proficiency levels in one-step and two-step deductions that respectively improved from 2.5% and 0.2% in Grade 9 to 63.3% and 42.6% in Grade 12. More recent findings by Atebe and Schäfer (2008) with a group of Nigerian and South African similarly showed that class inclusions of quadrilaterals among the investigated group from Grades 10-12 were almost completely absent.

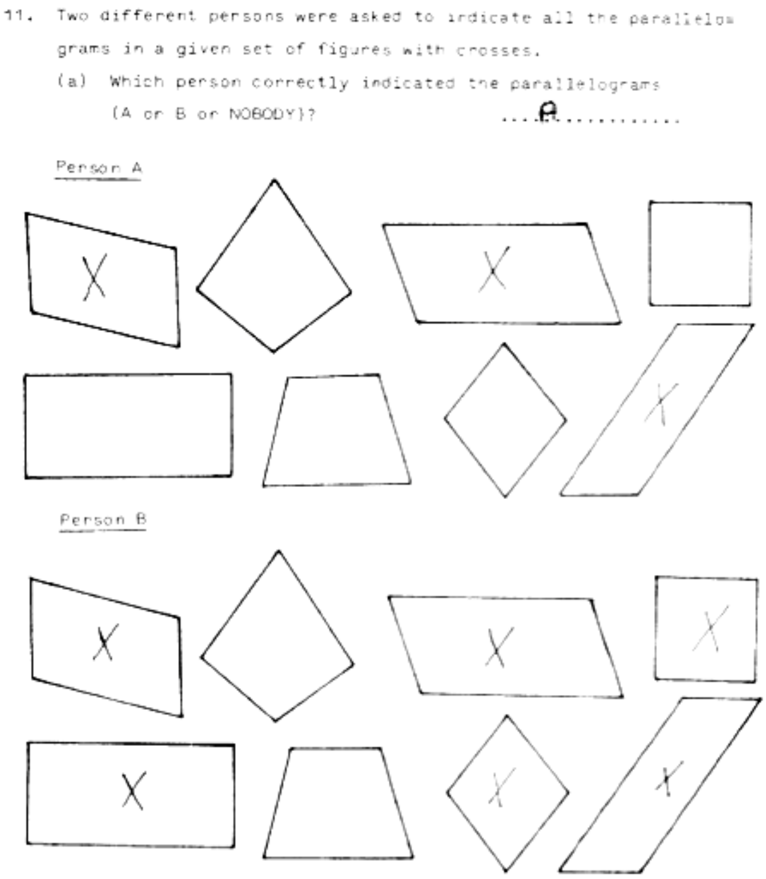

Figure 6. Testing class inclusion

Formal Van Hiele Level 3 definitions in textbooks are often preceded by an activity whereby students have to compare in tabular form various properties of the quadrilaterals, designed with the intention to assist students to see that a square, rectangle and rhombus have all the properties of a parallelogram, and that they therefore could be viewed as special cases. However, research reported in De Villiers (1994) shows that many students, even after doing tabular comparisons and other activities, if given the opportunity, still preferred to define quadrilaterals in *partitions*. (In other words, they would for example still prefer to define a parallelogram as a quadrilateral with both pairs of opposite sides parallel, but not all angles or sides equal).

For this reason, it seems that students should be allowed to formulate their own definitions irrespective of whether they are partitional or hierarchical. By then discussing and comparing in class the relative advantages and disadvantages of these two different ways of classifying and defining quadrilaterals (both of which are mathematically correct), students may be led to realize that there are certain advantages in accepting a hierarchical classification. For example, if students are asked to compare the following two definitions for the parallelograms, they might realize that the former is more **economical** than the latter:

hierarchical: A parallelogram is a quadrilateral with both pairs of opposite sides parallel.

partitional: A parallelogram is a quadrilateral with both pairs of opposite sides parallel, but not all angles or sides equal.

Clearly, partitional definitions are longer since they have to include additional properties to ensure the exclusion of special cases. Another advantage of a hierarchical definition for a concept is that all theorems proved for that concept then automatically apply to its special cases. For example, if we prove that the diagonals of a parallelogram bisect each other, we can immediately conclude that it is also true for rectangles, rhombi and squares. If however, we classified and defined them partitionally, we would have to prove separately in each case, for parallelograms, rectangles, rhombi and squares, that their diagonals bisect each other. Clearly to reproduce all these proofs is clearly very uneconomical. It seems clear that unless the role and function of a hierarchical classification is meaningfully addressed in class, many students will have difficulty in understanding why their intuitive, partitional definitions are not used.

Engaging students in defining geometric concepts like the quadrilaterals also provide valuable opportunity for students to learn how to construct counter-examples to incomplete or wrong definitions that they may come up with. For example, to be able to show that "*a kite is a quadrilateral with perpendicular diagonals*" is an incomplete definition to finding a quadrilateral with perpendicular diagonals that is not a kite.

One common difficulty students have in producing correct counterexamples to incomplete definitions is that they often try to refute a definition with a special case. For example, for the incorrect definition "*a rectangle is any quadrilateral with congruent diagonals*," some students will provide a square as a counterexample. But obviously a square is not a valid counterexample, because a square *is* a rectangle.

Therefore, students should already have developed a sound understanding of a hierarchical (inclusive) classification of quadrilaterals before being engaged in formally defining the quadrilaterals themselves (Casa & Gavin, 2009; Craine & Rubenstein, 1993). This development can be fostered by using interactive geometry software, figures created with flexible wire, or paper-strip models of quadrilaterals. Indeed, working with a group of five Grade 6 students using flexible wire and paper strip models, Malan (1986) found that they all were eventually able to successfully make hierarchical class inclusions of the quadrilaterals. In addition, he found that the language used for describing class inclusions was an important factor (e.g. calling a square a *special* rectangle).

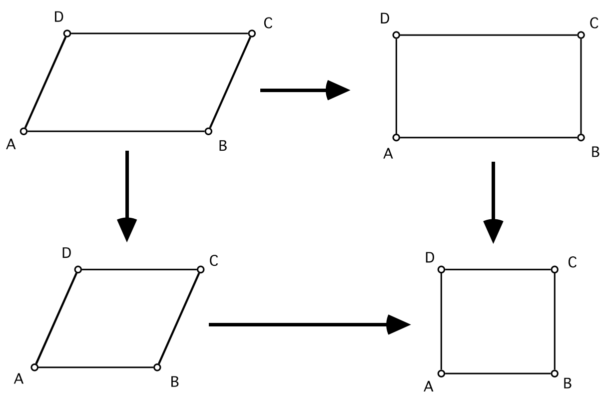

Figure 7. Dynamic transformation of parallelogram

Specifically, the dynamic nature of geometric figures constructed in dynamic software like *Sketchpad* may make the acceptance of a hierarchical classification of the quadrilaterals far easier at lower Van Hiele levels. For example, if students construct a quadrilateral with opposite sides parallel, then they will notice that they could easily drag it into the shape of a rectangle, rhombus or square as shown in Figure 7. In fact, it seems quite possible that at least some students would be able to accept and understand this even at Van Hiele Level 1 (Visualization), but further research into this particular area is needed. It is quite possible too that students' difficulties with hierarchical class inclusion is largely the result of traditional instructional practices, something already observed by Mayberry (1981:8) when she wrote: "It is conceivable that the observed levels are an artifact of the current curriculum or the instruction given to the students …".

The author has developed an experimental Java applet at http://math.kennesaw.edu/~mdevilli/quadclassify.html where the most common quadrilaterals are not introduced via formal definition, but simply introduced visually. Through guided dragging it is envisaged that a child at Van Hiele 1 may, for example, more easily develop a *dynamic concept image* of a rectangle as one that can change into a square when all its sides become equal. Teachers and researchers are invited to try out these activities and any feedback or reports are most welcome.

Construction and measurement

It should first be pointed out that certain kinds of construction activities (with dynamic geometry software or by pencil and paper) are inappropriate at Van Hiele Level 1. For example, someone was recently overheard at a conference commenting that she was unpleasantly dismayed at how difficult young children found the task of constructing a "*dynamic*" square with *Sketchpad*. However, if the children were still at Van Hiele Level 1, then it is not surprising at all—how can they construct a square if they do not yet know its properties (Level 2) and that some properties are sufficient and others not (that is, know the logical relationships between the properties—Level 3)?

In fact, at Van Hiele Level 1 it would appear to be far more appropriate to provide children with ready-made sketches of quadrilaterals in dynamic geometry software, which they can then easily manipulate and first investigate visually. Next, they could start using the measure features of the software to analyze the properties (and learn the appropriate terminology) to enable them to reach Level 2. Only then would it be appropriate to challenge them to construct such dynamic quadrilaterals themselves, thus assisting the transition to Level 3.

In other words, students who are predominantly at Van Hiele Level 2 cannot yet be expected to logically check their own descriptions (definitions) of quadrilaterals, but they should be allowed to do so by accurate construction and measurement. For example, students could evaluate the following attempted descriptions (definitions) for a rhombus by construction and measurement as shown in Figure 8:

(1) A rhombus is a quadrilateral with all sides equal.

(2) A rhombus is a quadrilateral with perpendicular, bisecting diagonals.

(3) A rhombus is a quadrilateral with bisecting diagonals.

(4) A rhombus is a quadrilateral with one pair of adjacent sides equal and both pairs of opposite sides parallel.

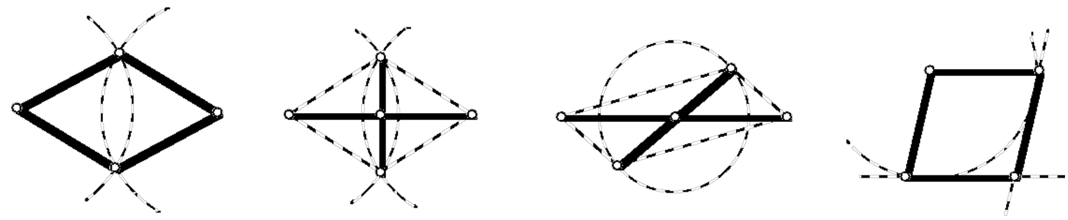

Figure 8. Construction and measurement

In the first example, students should construct a quadrilateral so that all four sides are equal, and then could notice that the diagonals always bisect each other perpendicularly, irrespective of how they drag it. This clearly shows that the property of "perpendicular bisecting diagonals" is a consequence of their constructing "all four sides equal." On the other hand, such testing also clearly shows when a description (definition) is incomplete (contains insufficient properties), as in the third example above.

Conceptually, constructions like these are extremely important for assisting the transition from Van Hiele Level 2 to Van Hiele Level 3. It helps to develop an understanding of the difference between a *premise* and *conclusion* and their *causal* relationship; in other words, of the logical structure of an "*if-then*" statement. For example, statement 4 could be rewritten by students as: "**If** a quadrilateral has one pair of adjacent sides equal and both pairs of opposite sides parallel, **then** it is a rhombus (that is, has all sides equal, perpendicular bisecting diagonals, and so on)". Smith (1940) reported marked improvement in students' understanding of "*if-then*" statements after letting them make constructions to evaluate geometric statements as follows:

> Pupils saw that when they did certain things in making a figure, certain other things resulted. They learned to feel the difference in category between the relationships they **put** into a figure - the things over which they had control - and the relationships which **resulted** without any action on their part. Finally the difference in these two categories was associated with the difference between the **given** conditions and **conclusion**, between the if-part and the then-part of a sentence.

Proof Phases in Geometry Education

According to the Van Hiele theory, for learning to be meaningful, students should become acquainted with, and explore, geometry content in phases which correspond to the Van Hiele Levels. A serious shortcoming of the Van Hiele theory, however, is that there is no explicit distinction between different possible functions of proof. For example, the development of deductive thinking appears first within the context of *systematization* at Van Hiele Level 3 (Ordering). Empirical research by De Villiers (1991) and Mudaly and De Villiers (2000) seem to indicate, however, that the functions of proof such as *explanation*, *discovery* and *verification* can be meaningful to students outside a systematization context, in other words, at Van Hiele Levels lower than Van Hiele Level 3, provided the arguments are of an intuitive or visual nature; for example, the use of symmetry or dissection.

From experience, it also seems that a prolonged delay at Van Hiele Levels 1 and 2 before introducing proof actually makes introducing proof later as a meaningful activity even more difficult. Examples of more fully developed proof and defining activities are available in De Villiers (2003, 2009). Defining quadrilaterals also provide an excellent context for introducing and developing students' understanding of necessary and sufficient conditions as discussed and illustrated in De Villiers et al (2009).

Concluding comments

The hierarchical, fixed order of progression through the Van Hiele levels (i.e. a pupil cannot be at level *n* without having passed through level *n-1*) have been statistically confirmed using Guttman analysis by several studies, for example, Mayberry (1981), Usiskin (1982) and De Villiers (1987). A comparative study by Smith and De Villiers (1989) of the Usiskin (1982) test and the University of Stellenbosch (1984) test further confirmed not only the hierarchical nature of the first three levels, but indicated that better classifications of students' thinking levels were obtained when more varied questions and more 'open-ended' items are used.

Pegg and Davey (1989) did a comparative study of the Van Hiele theory and the SOLO Taxonomy and found the descriptors of the latter more accurately described the geometric thinking levels of students. It is, however, still an open moot point whether such an achieved gain in 'accuracy' is worthwhile with the increased complexity of the Solo Taxonomy.

More research needs to be done on how using dynamic geometry software can enhance, or perhaps even impair, the development of geometric thinking. The use of dynamic geometry software with an experimental group of Malaysian students is reported in Idris (2009) to have contributed to them achieving higher Van Hiele levels than a control group who were taught traditionally without access to dynamic geometry.

A main concern of geometry education around the world is the continued poor level of geometric thinking among teachers themselves, and until this problem is adequately addressed, very little progress in the quality of geometry instruction is likely to be achieved. For example, Van Putten (2008) found in a post-test, that only 45% of pre-service FET (grades 10-12) teachers had reached Van Hiele Level 3 (though there was significant improvement from the pre-test).

Traditionally, the development of 'proof ability' is seen to occur from Van Hiele 3 Level onwards. Moreover, the Van Hiele model sees proof mainly as a means of 'verification', and it remains an open research question whether or not other functions of proof such as 'explanation' can be utilized and developed earlier at the visual and analytic Levels 1 and 2 respectively (see for example Mudaly & De Villiers, 2000; De Villiers, 2004). Can more explanatory visual-dissection proofs and arguments by symmetry (line, rotational, point) be developed and understood earlier by children?

Lastly, it seems one of the major outstanding research problems on the Van Hiele theory is the issue of hierarchical thinking (class inclusions). Is partition thinking the consequence of traditional geometry teaching strategies, and could hierarchical thinking be developed earlier at Van Hiele levels 1 and 2 through various strategies and using tools such as dynamic geometry software?

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