A Diagonal Property of a Rhombus Constructed from a Rectangle

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INTRODUCTION

For any given rectangle EBFD, a rhombus ABCD can be constructed (as shown in Figure 1) by ensuring that AB = AD. The easiest way to do this is to construct the perpendicular bisector of diagonal BD, since a rhombus has diagonals that bisect at right angles. The points A and C are located respectively where this perpendicular bisector cuts ED and BF.

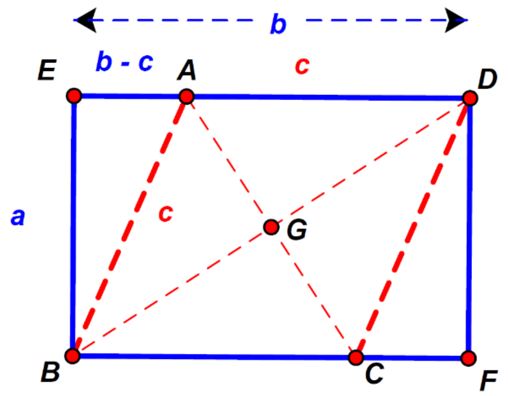

FIGURE 1: Rhombus constructed from a rectangle.

A dynamic geometry version of this construction, as well as two other constructions discussed later on, can be found at: http://dynamicmathematicslearning.com/metz-rhombus-construction.html

If the rectangle measures a by b with $a \leq b$, then the two diagonals of the rhombus, AC and BD, have an interesting relationship to the sides of the rectangle from which the rhombus was constructed:

$$\frac{AC}{BD} = \frac{a}{b}$$

PROOF AND DISCUSSION

The relationship can be shown very simply using similarity. Since triangles DEB and DGA are similar we have $\frac{a}{b} = \frac{AG}{GD} = \frac{\frac{1}{2}AC}{\frac{1}{2}BD} = \frac{AC}{BD}$ from which the result follows directly. Note that in the case where ABCD is a square, it follows from the construction that EBFD coincides with ABCD, and hence $\frac{AC}{BD} = \frac{a}{b} = 1$.

We can now also express the ratio of the diagonals in terms of the smaller angle $A\hat{D}C$ made by two sides of the rhombus. Since $\tan\left(\frac{A\hat{D}C}{2}\right) = \frac{a}{b}$, we have $\tan\left(\frac{A\hat{D}C}{2}\right) = \frac{AC}{BD}$. For example, if the smallest angle of the rhombus is 60°, then the ratio of the diagonals is $\tan 30° = \frac{1}{\sqrt{3}}$. Note that for a golden rectangle[4], in which $\frac{a}{b} = \frac{2}{1+\sqrt{5}} = \tan\left(\frac{A\hat{D}C}{2}\right)$, the smallest angle is about 63°. The rhombus associated with a golden rectangle is often called a 'golden rhombus', since its diagonals are in the golden ratio (see De Villiers, 2017).

[4] For a definition and more information about a 'golden rectangle', see: https://en.wikipedia.org/wiki/Golden_rectangle

It is worth noting that the described process (and resulting relationship) is reversible since from any given rhombus ABCD one can obtain a rectangle EBFD by dropping perpendiculars from B and D respectively to AD and BC extended.

OTHER RECTANGLE-RHOMBUS CONSTRUCTIONS

This 'dual' relationship between the sides of a rectangle and the diagonals of a rhombus is also nicely illustrated in Figure 2 as shown in De Villiers (2017). In the first figure the midpoint quadrilateral (generally called a 'Varignon parallelogram') of any given rectangle forms a Varignon rhombus with its diagonals in the same ratio as the sides of the rectangle. Conversely, as illustrated in the second figure in Figure 2, a Varignon rectangle is formed with its sides in the same ratio as the diagonals of the rhombus from which it is formed.

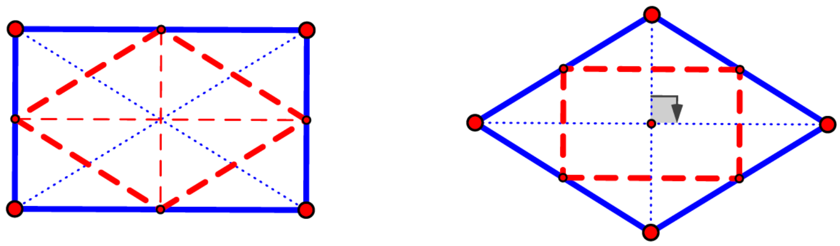

FIGURE 2: Varignon rhombi and rectangles.

Lastly, as proven in De Villiers (2017), starting with any given rectangle EFGH and its circumcircle as shown in Figure 3, and then constructing a rhombus ABCD with sides tangential to the circumcircle at the vertices of the rectangle, we find that the diagonals of ABCD are in the same ratio as the sides of EFGH. (Note that it follows directly from the symmetry of the rectangle EFGH that ABCD is a rhombus). Conversely, in any given rhombus, the 'touching points' of the incircle to its sides produce a rectangle with sides in the same ratio as the diagonals of the rhombus.

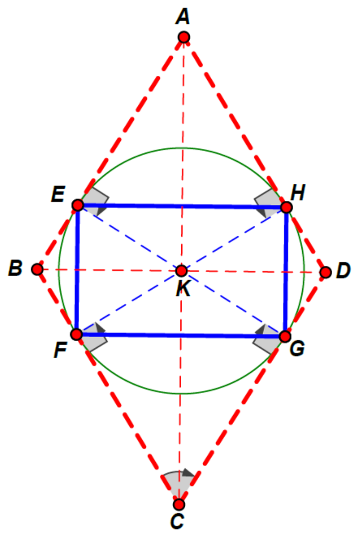

FIGURE 3: Tangential rhombus construction.

REFERENCES

De Villiers, M. (2017). An example of constructive defining: From a golden rectangle to golden quadrilaterals and beyond. *At Right Angles*[5], 6(1), March, pp. 64-69.

[5] Downloadable PDF copies of this mathematics education journal are available for free at: http://azimpremjiuniversity.edu.in/SitePages/resources-at-right-angles.aspx