Properties of Pythagorean quadrilaterals

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1. *Introduction*

There are many named quadrilaterals. In our hierarchical classification in [1, Figure 10] we included 18, and at least 10 more have been named, but the properties of the latter have only scarcely (or not at all) been studied. However, only a few of all these quadrilaterals are defined in terms of properties of the sides alone. Two well-known classes are the rhombi and the kites, defined to be quadrilaterals with four equal sides or two pairs of adjacent equal sides respectively. The orthodiagonal quadrilaterals are defined to have perpendicular diagonals, but an equivalent defining condition is quadrilaterals where the consecutive sides a, b, c, d satisfy $a^2 + c^2 = b^2 + d^2$. Then it is possible to prove that the diagonals are perpendicular and that no other quadrilaterals have perpendicular diagonals (see [2, pp. 13-14]). In the same way tangential quadrilaterals can be defined to be convex quadrilaterals where $a + c = b + d$. Starting from this equation, it is possible to prove that these and only these quadrilaterals have an incircle (since this equation is a characterisation of tangential quadrilaterals, see [3, pp. 65-67]).

In this paper we will study a few properties of convex quadrilaterals whose sides satisfy one of the quadratic conditions

$$a^2 + b^2 = c^2 + d^2 \qquad \text{or} \qquad a^2 + d^2 = b^2 + c^2. \tag{1}$$

These were called *Pythagorean quadrilaterals* in [4] together with the orthodiagonal quadrilaterals. The choice of name is hardly surprising. We will however exclude the orthodiagonal quadrilaterals from the class of Pythagorean quadrilaterals since we consider them to have sufficient differences to be two separate classes. In the same way trapezia and cyclic quadrilaterals are different classes although they have almost identical angle characterisations: two adjacent or two opposite supplementary angles respectively. Thus *we* define a Pythagorean quadrilateral to be *a convex quadrilateral where the consecutive sides* a, b, c, d *satisfy at least one of the two conditions* (1).

A few well-known special cases of Pythagorean quadrilaterals are squares, rectangles, rhombi, parallelograms, and kites. But also a quadrilateral where two opposite angles are both right angles is trivially a Pythagorean quadrilateral according to the Pythagorean theorem (more about such quadrilaterals in Section 5). There are however more irregular quadrilaterals in the class of Pythagorean quadrilaterals as well (see Figure 1).

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2. *Characterisations*

The first characterisation of Pythagorean quadrilaterals was proved in another way in [4].

Theorem 1: Let A' and C' be the feet of the perpendiculars from A and C respectively to the diagonal BD, and let B' and D' be the feet of the perpendiculars from B and D respectively to the diagonal AC in a convex quadrilateral $ABCD$. Then

$$AB^2 + BC^2 = CD^2 + DA^2 \Leftrightarrow BA' = DC' \Leftrightarrow BC' = DA'$$

and

$$AB^2 + DA^2 = BC^2 + CD^2 \Leftrightarrow AB' = CD' \Leftrightarrow AD' = CB'.$$

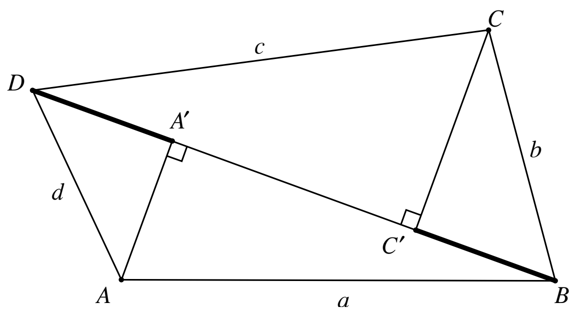

FIGURE 1: A Pythagorean quadrilateral

Proof: We prove the first condition; the second is proved in the same way. By the Pythagorean theorem we have $BA'^2 + AA'^2 = AB^2$, $BC'^2 + CC'^2 = BC^2$, $DC'^2 + CC'^2 = CD^2$ and $DA'^2 + AA'^2 = DA^2$ (see Figure 1). Thus

$$\begin{aligned} & AB^2 + BC^2 - CD^2 - DA^2 \\ = \ & BA'^2 - DA'^2 + BC'^2 - DC'^2 \\ = \ & (BA' + DA')(BA' - DA') + (BC' + DC')(BC' - DC') \\ = \ & BD(BC' + A'C' - DA') + BD(BC' - DA' - A'C') \\ = \ & 2BD(BC' - DA'). \end{aligned}$$

Hence

$$AB^2 + BC^2 = CD^2 + DA^2 \Leftrightarrow BC' = DA' \Leftrightarrow BA' = DC.$$

where the last equivalence follows by adding or subtracting $A'C'$ to both sides of $BC' = DA'$ depending on whether C' is closest to B or D.

Based on this theorem we conclude that a Pythagorean quadrilateral can also be defined as *a convex quadrilateral in which the feet of the normals to a diagonal through two opposite vertices have equal distances to each of the other two vertices*.

In the rest of this paper we use the notations $a = AB$, $b = BC$, $c = CD$ and $d = DA$ for the sides of a convex quadrilateral $ABCD$. The diagonals are denoted by $p = AC$ and $q = BD$. In a Pythagorean quadrilateral where $a^2 + b^2 = c^2 + d^2$, the diagonal p that divides the quadrilateral into two triangles with the sides a, b, p and c, d, p will be called the *main diagonal* (see Figure 2). Likewise, in a Pythagorean quadrilateral where $a^2 + d^2 = b^2 + c^2$, the diagonal q is the main diagonal that forms triangles having the sides a, d, q and b, c, q. Note that the second type of Pythagorean quadrilateral becomes the first on relabelling sides and vertices. Thus in the remainder of this paper we only consider the first type, but be aware that all theorems proved can be reinterpreted under the other possible labelling.

Theorem 2: Let h_B and h_D be the heights in the triangles BAC and DAC respectively to the diagonal AC in a convex quadrilateral $ABCD$ with consecutive sides a, b, c, d and no right angles. It is a Pythagorean quadrilateral with main diagonal p if, and only if,

$$\frac{\tan B}{h_B} = \frac{\tan D}{h_D}.$$

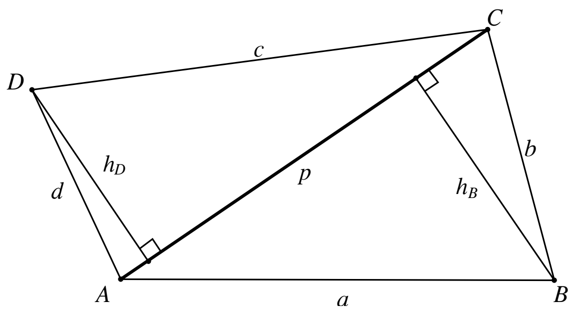

FIGURE 2: The heights to a main diagonal

Proof: Expressing twice the area of triangles BAC and DAC in two different ways respectively yields $ab \sin B = ph_B$ and $cd \sin D = ph_D$ (see Figure 2). Thus

$$\frac{ab}{cd} = \frac{h_B \sin D}{h_D \sin B} \tag{2}$$

which is valid in all convex quadrilaterals. Using the Law of Cosines in triangles BAC and DAC, we have that $a^2 + b^2 = c^2 + d^2$ is equivalent to

$$p^2 + 2ab \cos B = p^2 + 2cd \cos D \Leftrightarrow \frac{ab}{cd} = \frac{\cos D}{\cos B}. \tag{3}$$

Combining (2) and (3) yields

$$\frac{h_B \sin D}{h_D \sin B} = \frac{\cos D}{\cos B}$$

which is equivalent to the stated condition provided that $B \neq \frac{\pi}{2}$ and $D \neq \frac{\pi}{2}$.

An important property of Pythagorean quadrilaterals can be deduced from (3).

Theorem 3: The two angles opposite the main diagonal in a Pythagorean quadrilateral are always of the same type, i.e. they are both either acute, or right, or obtuse.

Proof: The cosine function is positive for acute angles, zero for right angles, and negative for obtuse angles. The quotient $\frac{ab}{cd}$ is always positive, so in the quotient $\frac{\cos D}{\cos B}$ on the right-hand side of (3), either both cosines must be positive or negative. Hence the opposite angles are either both acute or both obtuse. From the first part of (3) $ab \cos B = cd \cos D$, and we have that B is a right angle if, and only if, D is a right angle.

The third and last characterisation is about the measure of the angle between the diagonals in terms of the diagonal parts.

Theorem 4: In a convex quadrilateral with consecutive sides a, b, c, d and where no diagonal is bisected by the other, let the diagonals divide each other in parts w, x and y, z. It is a Pythagorean quadrilateral with main diagonal p if, and only if, the acute angle θ between the diagonals is given by

$$\cos \theta = \left| \frac{y - z}{w - x} \right|.$$

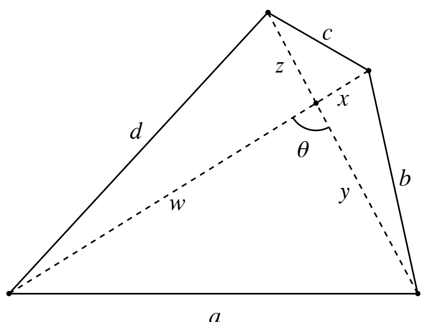

FIGURE 3: The diagonal parts

Proof: Using the Law of Cosines in the four non-overlapping subtriangles created by the diagonals (see Figure 3) yields that $a^2 + b^2 = c^2 + d^2$ is equivalent to

$$w^2 + y^2 - 2wy \cos \theta + y^2 + x^2 + 2xy \cos \theta$$

$$= x^2 + z^2 - 2xz\cos\theta + w^2 + z^2 + 2wz\cos\theta$$

$$\Leftrightarrow \qquad y^2 - z^2 = \cos\theta\left(wy - xy - xz + wz\right)$$

$$\Leftrightarrow \qquad (y + z)(y - z) = \cos\theta\,(w - x)(y + z). \tag{4}$$

This is equivalent to the formula in the theorem, where we put an absolute value to cover all cases (the other angle between the diagonals satisfies $\cos(\pi - \theta) = -\cos\theta$).

Corollary: One diagonal bisects the other diagonal in a Pythagorean quadrilateral if, and only if, it is a parallelogram or a kite.

Proof: From (4) we get that $y = z$ if, and only if, $w = x$ or $\theta = \frac{\pi}{2}$. The first possibility gives a parallelogram and the second a kite. The converses are well-known properties of these quadrilaterals.

3. *Area*

The area of a convex quadrilateral can be calculated given the length of the four sides a, b, c, d and the two diagonals p, q. The formula, which was originally derived independently by the two German mathematicians von Staudt and Bretschneider in 1842, is

$$K = \tfrac{1}{4}\sqrt{4p^2q^2 - (a^2 - b^2 + c^2 - d^2)^2}. \tag{5}$$

Two modern proofs were given in [5]. This formula can be simplified in a Pythagorean quadrilateral.

Theorem 5: A Pythagorean quadrilateral with consecutive sides a, b, c, d and diagonals p, q has the area

$$K = \tfrac{1}{2}\sqrt{p^2q^2 - (a^2 - d^2)^2} = \tfrac{1}{2}\sqrt{p^2q^2 - (c^2 - b^2)^2}$$

when p is the main diagonal.

Proof: Using the condition $a^2 + b^2 = c^2 + d^2$, or rather its equivalent version $a^2 - d^2 = c^2 - b^2$, in (5) directly yields the two formulas after factoring out a 4 under the square root.

The next theorem gives area formulas in terms of two adjacent sides, their intermediate angle, and its opposite angle.

Theorem 6: A Pythagorean quadrilateral $ABCD$ with consecutive sides a, b, c, d and no right angles has the area

$$K = \frac{ab\sin(B + D)}{2\cos D} = \frac{cd\sin(B + D)}{2\cos B}$$

when p is the main diagonal.

Proof: As in the proof of Theorem 2, $a^2 + b^2 = c^2 + d^2$ is equivalent to

$$\frac{ab}{cd} = \frac{\cos D}{\cos B}. \tag{6}$$

By dividing a quadrilateral into two triangles using a diagonal, we have that the area of any convex quadrilateral satisfies $2K = ab \sin B + cd \sin D$. Hence

$$2K = cd\frac{\cos D}{\cos B} \sin B + cd \sin D = cd\left(\frac{\sin B \cos D + \cos B \sin D}{\cos B}\right).$$

The addition formula for sine now yields the second formula and the first follows directly by using (6) again. The exception for right angles is due to the cosines in the denominators as well as the sines in the numerators, which in that case are both zero.

There is another trigonometric area formula which does not have any angle restrictions.

Theorem 7: A Pythagorean quadrilateral *ABCD* with consecutive sides a, b, c, d and main diagonal p has area

$$K = \frac{1}{2}\sqrt{(ab + cd)^2 - 4abcd \cos^2\left(\frac{A + C}{2}\right)}.$$

Proof: Using the identity

$$\left(a^2 - b^2 + c^2 - d^2\right)^2 - \left(a^2 + b^2 - c^2 - d^2\right)^2 = -4\left(a^2 - d^2\right)\left(b^2 - c^2\right)$$

in (5), we get that the area of any convex quadrilateral is given by

$$K = \tfrac{1}{4}\sqrt{4p^2q^2 - (a^2 + b^2 - c^2 - d^2)^2 + 4(a^2 - d^2)(b^2 - c^2)}. \tag{7}$$

Bretschneider's generalisation of Ptolemy's theorem (discovered in 1842 and rediscovered in 2001, see the proof in [6]) states that in all convex quadrilaterals

$$p^2q^2 = a^2c^2 + b^2d^2 - 2abcd \cos(A + C). \tag{8}$$

Eliminating the product of the diagonals and inserting $a^2 + b^2 = c^2 + d^2$ in (7) yields

$$K = \tfrac{1}{4}\sqrt{4\left(a^2c^2 + b^2d^2 - 2abcd \cos(A + C)\right) + 4(a^2 - d^2)(b^2 - c^2)}.$$

$$= \tfrac{1}{2}\sqrt{a^2b^2 + c^2d^2 - 2abcd \cos(A + C)}$$

$$= \tfrac{1}{2}\sqrt{(ab + cd)^2 - 2abcd\left(1 + \cos(A + C)\right)}.$$

The last step is to use the half-angle formula for cosines.

The area of a convex quadrilateral with diagonals p and q is (see [7])

$$K = \tfrac{1}{2}pq \sin\theta, \tag{9}$$

where θ is the angle between the diagonals. Thus the area is half the product of the diagonals if, and only if, the diagonals are perpendicular. Comparing formula (9) with the ones in Theorem 5, we conclude that a Pythagorean quadrilateral is *orthodiagonal* if, and only if, at least one pair of adjacent sides are equal. But then the other pair is also equal according to the defining conditions (1), so a Pythagorean quadrilateral is orthodiagonal if, and only if, it is a kite.

In [2, p. 19] we proved that the bimedians (the two line segments connecting the midpoints of opposite sides) are equal if, and only if, the diagonals are perpendicular. Hence a Pythagorean quadrilateral has equal bimedians if, and only if, it is a kite.

4. *Incircle and excircle*

An *incircle* is a circle inside the quadrilateral that is tangent to all four sides, and an *excircle* is a circle outside the quadrilateral that is tangent to the extensions of all four sides. Quadrilaterals having an incircle or an excircle are called *tangential* and *extangential* respectively. Some similar features of these quadrilaterals were discussed in [8].

The two defining conditions (1) for a Pythagorean quadrilateral can be merged into one condition as

$$|a^2 - c^2| = |b^2 - d^2|.$$

Factorising this yields

$$(a + c)|a - c| = (b + d)|b - d|.$$

It is well-known that a convex quadrilateral has an incircle if, and only if, its sides satisfy $a + c = b + d$ according to Pitot's theorem. In [8, p. 64], we noted that a convex quadrilateral has an excircle if, and only if, $|a - c| = |b - d|$. Thus we conclude that a Pythagorean quadrilateral has an incircle if, and only if, it has an excircle. Combining the incircle and excircle conditions, we get that the sides satisfy $a = b$ and $c = d$ or $a = d$ and $b = c$. In both cases the quadrilateral is a kite. Hence a Pythagorean quadrilateral has an incircle if, and only if, it is a kite, and the same is true for an excircle. Together with the defining conditions of a kite, this means that the kites (including their special cases rhombi and squares) are the only quadrilaterals that are both tangential and extangential.

We summarise the conclusions from this and the previous section regarding kites in the following theorem.

Theorem 8: A Pythagorean quadrilateral has any of

(i) two adjacent equal sides

(ii) perpendicular diagonals

(iii) an area that is half the product of the diagonals

(iv) equal bimedians

(v) an incircle

(vi) an excircle

if, and only if, it is a kite.

5. *Circumcircle*

We define a *right Pythagorean quadrilateral* to be a Pythagorean quadrilateral where both angles opposite the main diagonal are right angles, see Figure 4. (In [9, pp. 154-155] Michael de Villiers calls a quadrilateral with two opposite right angles a 'right quadrilateral'.) These quadrilaterals appear quite often in problem solving. One example: In a triangle ABC, the internal and external angle bisector to each vertex angle are perpendicular. Since the incentre I and the three excentres J_A, J_B and J_C lie on the intersections of internal or external angle bisectors, there are three right Pythagorean quadrilaterals associated with the four tritangent circles to every triangle. These are the quadrilaterals $AIBJ_C$, $BICJ_A$ and $CIAJ_B$.

We have the following necessary and sufficient condition.

Theorem 9: A Pythagorean quadrilateral is cyclic if, and only if, it is a right Pythagorean quadrilateral.

Proof: It is well known that a convex quadrilateral is cyclic, that is, it has a circumcircle, if, and only if, it has two opposite supplementary angles. In a right Pythagorean quadrilateral two opposite angles are both right angles and thus supplementary, so it is cyclic. Conversely, in a cyclic Pythagorean quadrilateral there are a pair of opposite angles of the same sort (according to Theorem 3) that are supplementary, so they must both be right angles since they can neither be both acute nor both obtuse.

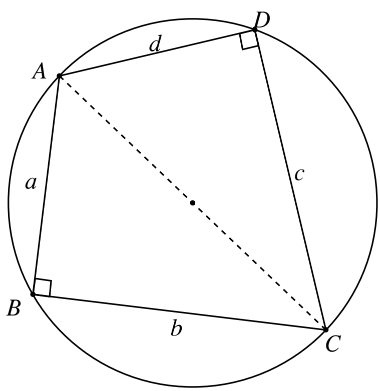

FIGURE 4: A right Pythagorean quadrilateral

Calculating the area of a right Pythagorean quadrilateral is an easy task since it can be divided by a diagonal into two right triangles. A right

Pythagorean quadrilateral *ABCD* with consecutive sides a, b, c, d and where B and D are right angles has the area

$$K = \frac{ab + cd}{2}.$$

This formula is also a corollary to Theorem 7, where the cosine term is zero since $A + C = \pi$.

The main diagonal of a right Pythagorean quadrilateral is a diameter of the circumcircle, see Figure 4. Thus using the Pythagorean theorem directly yields formulas for the circumradius. A right Pythagorean quadrilateral *ABCD* with consecutive sides a, b, c, d and where B and D are right angles has the circumradius

$$R = \tfrac{1}{2}\sqrt{a^2 + b^2} = \tfrac{1}{2}\sqrt{c^2 + d^2}.$$

The length of the second diagonal can then be calculated using Ptolemy's theorem.

6. *Comparison with orthodiagonal quadrilaterals*

Although we defined orthodiagonal and Pythagorean quadrilaterals to be separate classes of quadrilaterals due to their differences, they have in fact a few similarities, and these become more prominent in the special cases when they have an incircle or a circumcircle. We conclude this paper by making comparisons on their similar properties.

First of all, the characteristic properties of the sides are strikingly similar. In an orthodiagonal quadrilateral, we have $a^2 + c^2 = b^2 + d^2$, whereas in a Pythagorean quadrilateral it holds that $a^2 + b^2 = c^2 + d^2$ (or $a^2 + d^2 = b^2 + c^2$).

When it comes to the area, there is in fact a formula very similar to the one in Theorem 7 for orthodiagonal quadrilaterals.

Theorem 10: An orthodiagonal quadrilateral *ABCD* with consecutive sides a, b, c, d has area

$$K = \frac{1}{2}\sqrt{(ac + bd)^2 - 4abcd\cos^2\left(\frac{A + C}{2}\right)}.$$

Proof: By adding and subtracting $2abcd$ to the right hand side of (8), we have that in a convex quadrilateral

$$p^2q^2 = (ac + bd)^2 - 2abcd\left(1 + \cos(A + C)\right)$$
$$= (ac + bd)^2 - 4abcd\cos^2\left(\frac{A + C}{2}\right).$$

Now the formula follows at once, since an orthodiagonal quadrilateral has the area $K = \tfrac{1}{2}pq$.

We have previously proved in Theorem 8(v) that a Pythagorean quadrilateral has an incircle if, and only if, it is a kite. There is a similar condition for an orthodiagonal quadrilateral.

Theorem 11: An orthodiagonal quadrilateral has an incircle if, and only if, it is a kite.

Proof: We solve the two simultaneous equations $a^2 + c^2 = b^2 + d^2$ and $a + c = b + d$ that characterise a convex quadrilateral to be orthodiagonal and tangential respectively. Rewriting them as $(a + b)(a - b) = (d + c)(d - c)$ and $a - b = d - c$ we have the two cases that either $a = b$ and thus $d = c$, or $a \neq b$, and then the first equation is simplified to $a + b = d + c$. Combining this with $a - b = d - c$, the solution in the second case is $a = d$ and $b = c$. In either case the quadrilateral is a kite.

In the previous section we stated formulas for the area K and circumradius R of cyclic (i.e. right) Pythagorean quadrilaterals. For the cyclic orthodiagonal quadrilateral there are the similar formulas

$$K = \frac{ac + bd}{2}$$

and

$$R = \tfrac{1}{2}\sqrt{a^2 + c^2} = \tfrac{1}{2}\sqrt{b^2 + d^2}.$$

The area formula follows directly when inserting $A + C = \pi$ in Theorem 10. Another short proof is by inserting Ptolemy's theorem $pq = ac + bd$ (valid in cyclic quadrilaterals) in the formula $K = \tfrac{1}{2}pq$ for the area of an orthodiagonal quadrilateral.

To derive the formulas for the circumradius, first note that in all orthodiagonal quadrilaterals, we have with the notation in Figure 5 that

$$a^2 + c^2 = w^2 + x^2 + y^2 + z^2 = b^2 + d^2$$

according to the Pythagorean theorem applied four times, where w, x, y, z are the diagonal parts. Since

$$u = \left|\frac{y + z}{2} - z\right| = \frac{|y - z|}{2},$$

then by the Pythagorean theorem in triangle BOQ we get

$$\left(\frac{w + x}{2}\right)^2 + \left(\frac{y - z}{2}\right)^2 = R^2.$$

Whence

$$4R^2 = w^2 + 2wx + x^2 + y^2 - 2yz + z^2 = a^2 + c^2$$

where $wx = yz$ according to the intersecting chords theorem, completing the derivation.

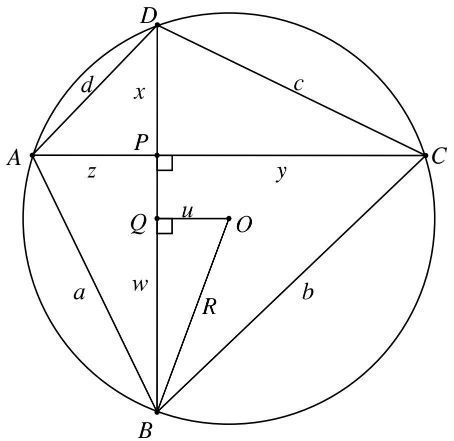

FIGURE 5: A cyclic orthodiagonal quadrilateral

It was claimed in [10, p. 26] that $w^2 + x^2 + y^2 + z^2 = \mathscr{D}^2$, where $\mathscr{D}$ is the diameter of the circle, is a new result from 2004. This is however not true. It is in fact Theorem 11 from Archimedes' *Book of Lemmas*, which states that if two chords *AB* and *CD* of a circle intersect at right angles at *P*, then

$$AP^2 + BP^2 + CP^2 + DP^2 = \mathscr{D}^2.$$

A different derivation of this equality from the one we gave can be found in [11, pp. 104-105].

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