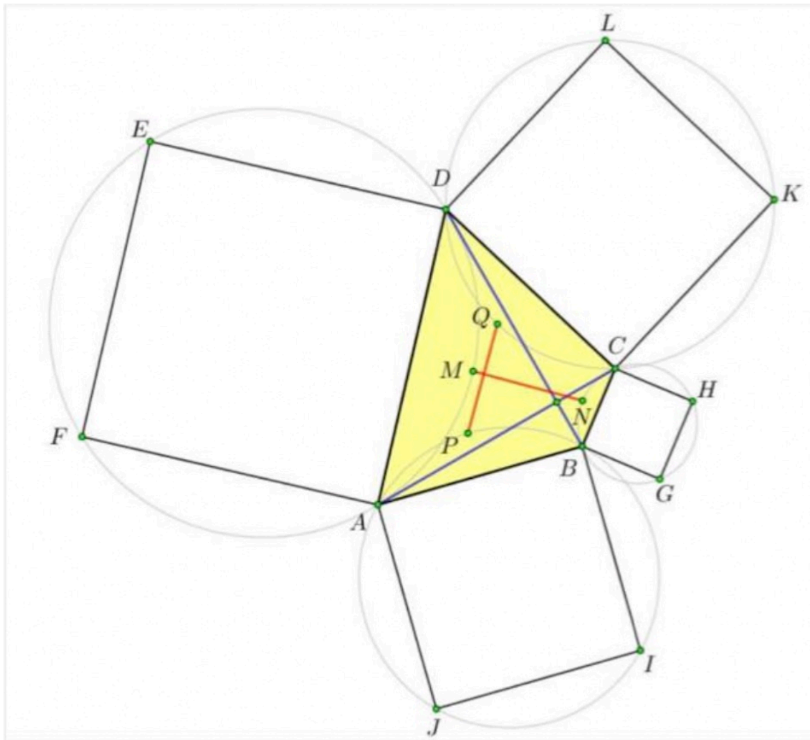
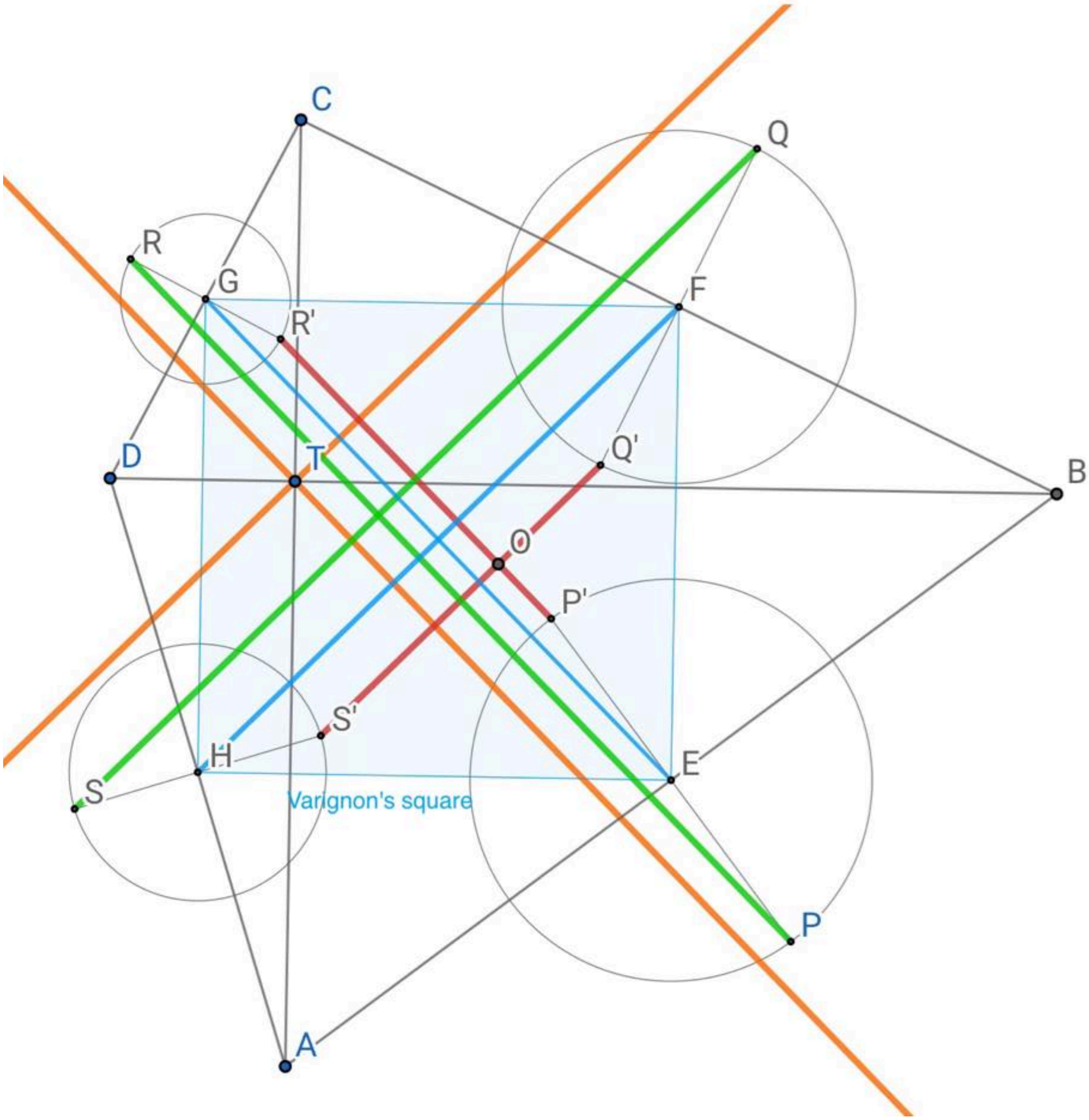
Jean-Pol Coulon from Tournai, Belgium posted the following interesting related problem and proof on 17/18 October 2024 in the *Romantics of Geometry* Facebook Group in relation to an earlier post by Ercole Suppa, RG 13475, about a quadrilateral that is both orthodiagonal and equidiagonal.



- ⋆ $ABCD$ quadrilateral with $AC = BD$ and $AC \perp BD$
- ⋆ $M =$ midpoint of arc AD of $\odot(ADEF)$
- ⋆ $N =$ midpoint of arc BC of $\odot(BGHG)$
- ⋆ $P =$ midpoint of arc AB of $\odot(ABIJ)$
- ⋆ $Q =$ midpoint of arc CD of $\odot(CDLK)$

Prove: $MN \perp PQ$ and $MN = PQ$

On the diagram below, all the segments (exple RR') perpendicular to the midpoints E, F, G and H of the sides (exple G midpoint of CD) of the orthodiagonal quadrilateral ABCD (congruent internal diagonals AC and BD) are in the same ratio wrt their resp. side, realizing virtually similar isosceles triangles outside and inside the quadrilateral, symmetrical wrt each side (exple CRD and CR'D, APB and AP'B).



Obviously, whatever the 8 similar isosceles, by construction and by Thales in 4 trapezoids (exple RR'P'P) similarly cut by an horizontal medial secant (exple GE), the constructed segments (exple PP' and RR') are parallel to

• the diagonals of the Varignon's square (exple GE) having 4 sides parallel to the orthogonal diagonals of the quadrilateral, diagonals being also the horizontal medians of the constructed trapezoids,

• the (orange) angle bisectors of the diagonals of the orthodiagonal quadrilateral.

Conclusion follows.
By construction of the solution, we don't need 4 similar internal isosceles triangles and 4 similar external triangles. We just need pairs of similar opposite isosceles triangles in the special orthodiagonal quadrilateral, either internal or external. Therefore the internal and the external isosceles triangles on a same side don't have to be similar, but have to be similar with the corresponding opposite triangle, the internal with the other internal, and the external with the other external triangle.
If the 2 diagonals are not congruent, the quadrilateral of Varignon is a rectangle (diagonals not orthogonal), and the properties are lost.
The same reasoning proves the lemma RG13475 of Ercole Suppa, just a special case of the exercise presented here.