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Place Kicking Locus in Rugby

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The place kicking problem in rugby is a lovely problem that I first came across in a presentation by a well loved doyen of mathematics education in South Africa, namely, Tickey de Jager (then teacher at Rondebosch Boys High School, now retired). The problem also appears in his book (De Jager, 1996: 158-161), as well as in Picciotto (1999: 19-24) where it is contextualised (a little differently) within soccer. With the upcoming World Cup Rugby it seems appropriate to revisit this problem as an example of the application of mathematics to the real world.

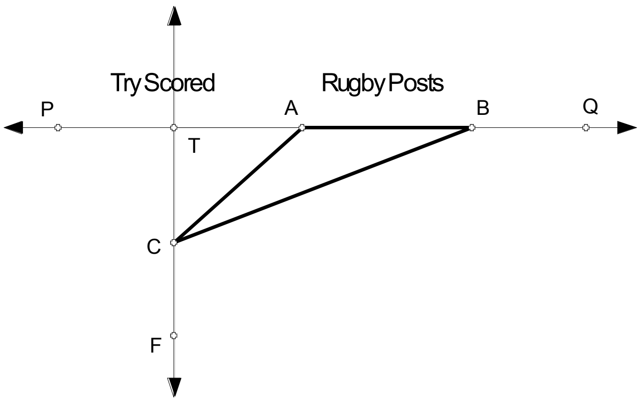

Figure 1

The problem can be stated as follows with reference to Figure 1. Suppose a try is scored at T, then the kicker can kick to the posts from anywhere along line TF perpendicular to the tryline PQ. Where is the best place to kick from?

Many kickers seem to think that the further they go back (within their kicking limit) the better, but that is not necessarily the case. If we ignore the actual distance to be kicked, the best position is clearly where the angle ACB subtended by the posts along TF is a maximum. But how can this maximum angle be found?

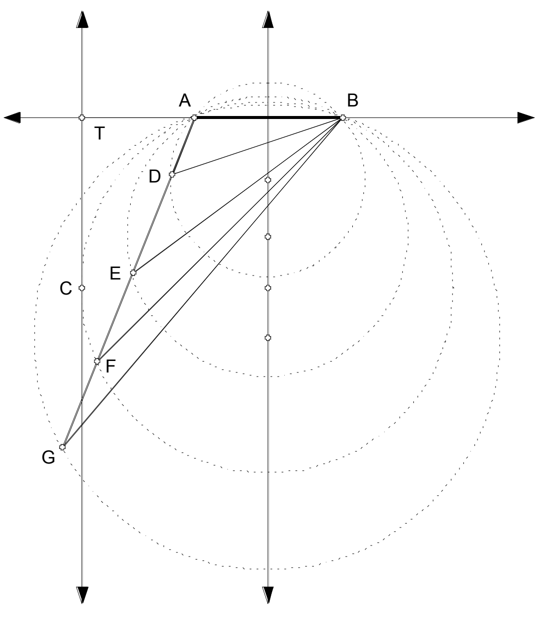

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Figure 2

With *Sketchpad* we can, of course, make a dynamic sketch, measure the angle ACB, and find its maximum value simply by dragging C along line TF. However, the solution can also be found elegantly by imagining a series of expanding circles with AB as chord (see Figure 2). Note that in each of the expanding circular arcs all the inscribed angles are equal (inscribed angles on the same chord). Furthermore, note that these sets of equal angles on the circular arcs decrease as the circles get larger (from the exterior angle theorem for a triangle, angle ADB > angle AEB > angle AFB > angle AGB). It therefore follows that the optimal solution would be found where one of the expanding circles touches the line TF (i.e. is tangent to it).

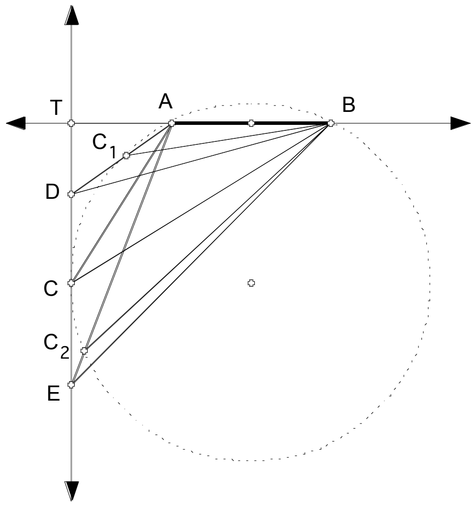

Figure 3

Another slightly different way to see that this is the optimal solution is shown in Figure 3. Suppose any two points D and E are respectively chosen on either side of C, then it follows as before with the exterior angle theorem that

$\angle ACB = \angle AC_1B = \angle AC_2B$

is always greater than any angle ADB or angle AEB.

It is, however, now interesting to ask what is the locus (path) of the optimal points C as T moves along the tryline. I'm indebted for this interesting further extension to the place kicking problem to Troy Jones, a high school mathematics teacher from Canada, who mentioned it to me at the NCTM congress in April 1999 in San Francisco. Using *Sketchpad*'s trace or locus construction facility this locus can easily be produced as shown in Figure 4. Clearly it appears to be part of a conic, but which of the conics?

By reflecting point C and the tangent circle as shown in Figure 5, it becomes clear that the locus is a hyperbola. In order to prove that it is indeed a hyperbola, we can introduce coordinate axes as shown with AO = -*a* and BO = *a*. Consider a point F on the locus with coordinates (x; y). Then DE = $x + a$ and EF = $x - a$. Since DF is a diameter, angle DBF = 90°, and from the similarity of triangles DEB and BEF it follows that:

$$\frac{y}{x-a} = \frac{x+a}{y}$$
$$\Leftrightarrow y^2 = (x-a)(x+a) = x^2 - a^2$$
$$\Leftrightarrow a^2 = x^2 - y^2$$
$$\Leftrightarrow 1 = \frac{x^2}{a^2} - \frac{y^2}{a^2}$$

which is the standard algebraic form for a rectangular hyperbola.

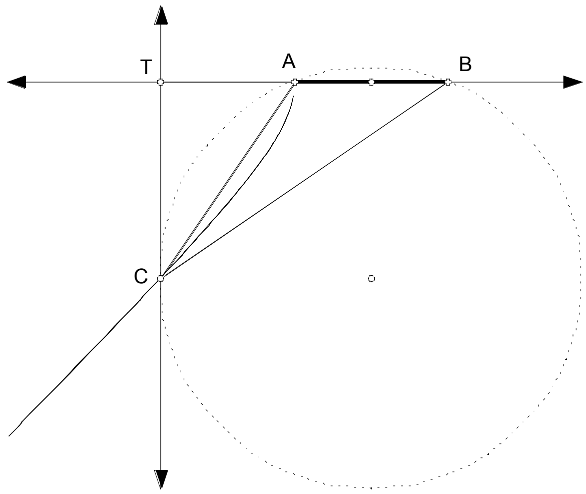

Figure 4

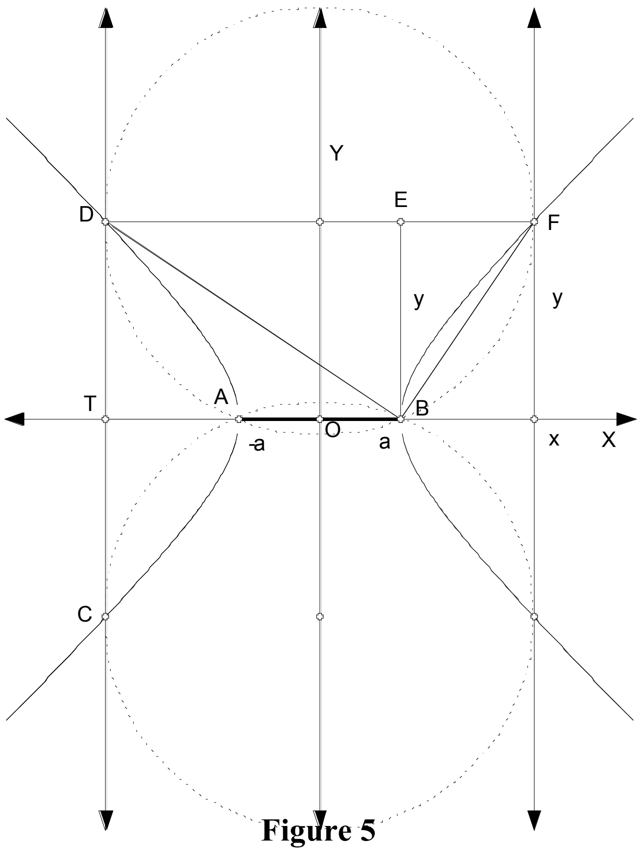

Figure 5

It is certainly tempting to consider painting these hyperbolic curves onto rugby fields so that all a kicker would have to do is to move from the tryline perpendicularly until he meets the curve. However, how well does this mathematical model fit with reality? As can clearly be seen from either Figure 4 or 5, as a try approaches the posts, the hyperbola rapidly approaches an upright post. This means that if a try is scored very close to the posts, the largest (optimal) angle is also very close to the posts. Clearly, this is impractical as the ball must also be kicked over the crossbar, and if one is too close to the posts, the angle of elevation becomes high, increasing the difficulty of the kick. (In the extreme case, consider a try being scored directly behind one of the upright posts, and where the optimal position is given at the base of the upright!). It would therefore appear that our model is only likely to give reasonably good results for tries scored further from the posts.

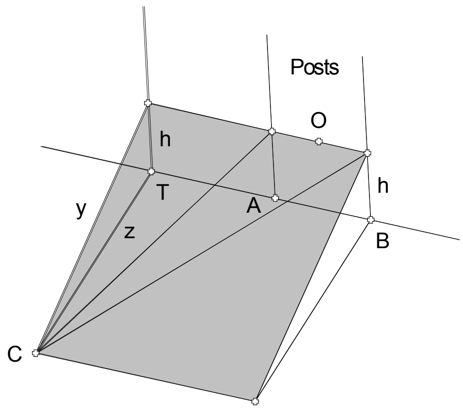

Figure 6

From the preceding discussion, it should now also be clear that our model is based on an over-simplification of reality, as we have not accounted at all for the height of the crossbar above the ground. (Basically we had assumed it was at ground, which is perhaps not too bad if one is far away, but the closer one is, the more important it becomes). In fact, as shown in Figure 6, the required tangent circles lie on a series of planes running through the crossbar with the optimal positions located where these planes cut the ground. The relationship between the optimal distance y on these inclined planes to the corresponding distance z on the ground is given by the theorem of Pythagoras. If we let the height of the crossbar be h, then

$$z^2 = y^2 - h^2 = x^2 - a^2 - h^2$$

which gives us an improved hyperbolic curve on the ground. To simplify matters, let's assume $h = a$, and draw both the original hyperbola (C) and the improved one (C') on the same axes with *Sketchpad* as shown in Figure 7. As can be seen, this adapted hyberbola approaches the original one as x increases, but deviates from the original as x decreases (when the try is scored closer to the posts). Note that whereas the original hyperbola cuts the x-axis at $-a$ and a respectively, the adapted one in Figure 7 cuts it at

$$\pm\sqrt{2}a$$

(assuming $h = a$). As with the original hyperbola, the adapted hyperbola becomes unreliable as it approaches these values.

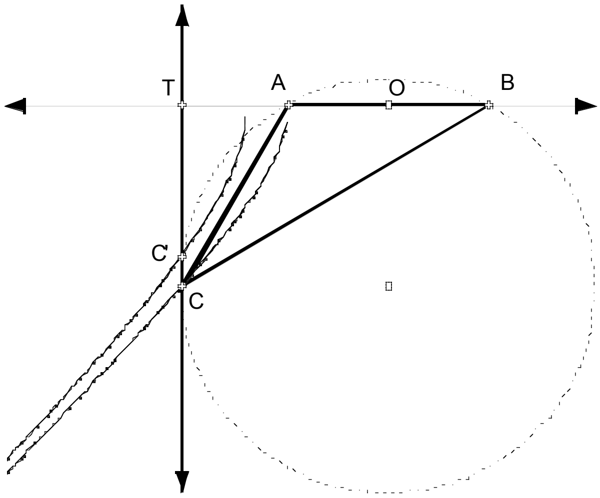

Figure 7

It also seems natural to ask: where is the optimal position to kick from if a try is scored inside the goal posts, rather than outside? If we ignore the height of the crossbar (the angle of elevation), the largest kicking angle is obviously on the tryline (with an angle of 180°). Again this is completely impractical as it would be impossible to kick the ball over from a position directly below the crossbar. Therefore, the kick should not be taken too close to the tryline (as the angle of elevation is then too high), but also not too far away (as the kicking angle then becomes too small). The kicker therefore needs to find some balance between the angle of elevation and the kicking angle, depending on his own preferences or strengths.

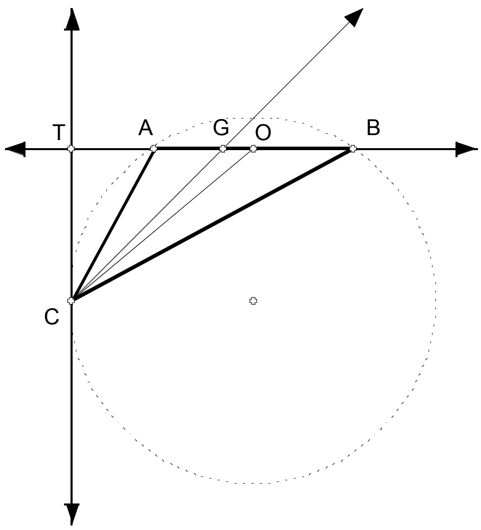

Figure 8

Lastly, a final question: which is the best direction to kick in from the optimal position? De Jager (1996:59) argues that the kicker should aim along the angle bisector of angle ACB shown by line CG in Figure 8 (and then goes on to prove that the angle bisector makes an angle of 45º with the tryline AB; i.e. angle TGC = 45º). However, as shown in Figure 8 the optimal direction appears to be along the median CO instead, which would imply that the ball passes directly over the centre of the crossbar (and allows for equal amount of error on both sides). Another assumption not previously stated should also be noted here, namely, that it is assumed that the kicker kicks in a straight line. (More precisely, it is assumed that the parabolic path of the ball lies in a vertical plane so that seen from directly above, it appears like a straight line). However, in reality many kickers kick the ball in a curve rather than a straight line. Typically, a right-footed kicker would curve the ball from right to left, and a left-footed kicker would curve it from left to right. For example, a right-footed kicker may kick a ball from C in Figure 8 to initially curve outside CB, but eventually curling in to cross the bar at its midpoint O. (Therefore the parabolic path of the ball does not necessarily lie in a vertical plane, but could instead lie in a vertically curved surface.)

The place kicking problem also illustrates how real world problems are often far more complex than they may appear at first sight, and that our mathematical models usually rely on certain simplifying assumptions which may not correspond precisely with reality. However, trying to account for every variable might increase the mathematical complexity to such a degree that we are often satisfied with an approximate, but simpler model. In fact, many real world situations are so complex, that it is almost impossible to consider all the variables concerned.

Note

Sketchpad is available in Southern Africa from *Dynamic Mathematics Learning,* 32 Belfry Towers, Doonside 4141. Tel: 031-9035109 (h); or 083-6561396 (cell).
E-mail: dynamiclearn@mweb.co.za
Also visit: http://go.keycurriculum.com/sketchpad_trial.html to download a free demonstration copy of *Sketchpad 5*.

References

De Jager, T. (1996). **More than Just Maths**. Pietermaritzburg: Shuter & Shooter.
Picciotto, H. (1999). **Geometry Labs**. Emeryville, CA: Key Curriculum Press.