THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

SENIOR SECOND ROUND 2022

Solutions

1. **Answer** 026
   Since multiplication and division must be done first, the value is $3 - 5 + (8 \div 2 \times 7) = 3 - 5 + 28 = 26$.

2. **Answer** 084
   The positive multiples of 6 less than 100 are $1 \times 6 = 6, 2 \times 6 = 12$, and so on, up to $16 \times 6 = 96$. There are 16 multiples of 6, leaving 84 numbers that are not multiples of 6. Alternatively, one could divide 100 by 6 which is 16 and ignore the remainder.

3. **Answer** 120
   The height of the snowman is the sum of the diameters of the spheres, which is twice the sum of their radii, so the height in centimetres is $2 \times (10 + 20 + 30) = 2 \times 60 = 120$.

4. **Answer** 015
   The average or mean of 5 and 9 is 7, so the average of 5 and x must be 10, that is, $\frac{1}{2}(5 + x) = 10$, so $5 + x = 20$ and $x = 15$. Check by showing that $\frac{1}{2}(9 + x) = 12$.

5. **Answer** 024
   Since $600 = 2^3 \times 3 \times 5^2$, it follows that every factor of 600 must be of the form $2^i 3^j 5^k$, where $0 \le i \le 3$, $0 \le j \le 1$, and $0 \le k \le 2$. This gives a total of $4 \times 2 \times 3 = 24$ different factors.

6. **Answer 035**
   Suppose the price of a pink pill is Rx, so the price of a blue pill is R$(x + 1)$. The cost in rands for one day is therefore $(x + 1) + 2x = 3x + 1$, and for ten days is $10(3x + 1) = 520$. This gives $3x = 51$, so $x = 17$ and $x + (x + 1) = 35$.

7. **Answer** 130
   Produce (or extend) BC to cut DE at F. Then $C\widehat{F}E = 90° + 40° = 130°$ (exterior angle of triangle DCF), but $C\widehat{F}E = C\widehat{B}A = x$ (alternate angles), so $x = 130°$.

8. **Answer** 012
   The sum of the interior angles of an n-sided polygon is $(n - 2)180°$, so each angle of a regular n-gon is $(1 - \frac{2}{n})180°$. For a regular pentagon ($n = 5$), this gives $108°$, and for a regular hexagon ($n = 6$) it is $120°$. The required angle is therefore $120° - 108° = 12°$.

9. **Answer** 484
   The combined area of the four triangles at the corners is equal to the area of one square, and the combined area of the two triangles in the middle is also equal to that of one square. The total rectangle therefore has the same area as four squares, that is, $4 \times 121 = 484$.

10. **Answer** 005
    The combined speed of the cars is 90 km/h, so the time taken to cover 450 km is $450 \div 90 = 5$ hours.

11. **Answer** 036
If the base diameter is increased by 25%, then the new diameter is $1 + \frac{25}{100} = \frac{5}{4}$ times the old diameter. The new base area is therefore $(\frac{5}{4})^2 = \frac{25}{16}$ times the old area. To maintain the same volume, the new height must be $\frac{16}{25}$ times the old height, so the percentage reduction is $(1 - \frac{16}{25}) \times 100\% = \frac{9}{25} \times 100\% = 36\%$.

12. **Answer** 020
The percentage of learners who brought an orange is $100 - (20 + 35) = 45\%$. If there are n learners in the class, then $9 = \frac{45}{100}n = \frac{9}{20}n$, so $n = 20$.

13. **Answer** 044
The base length of the triangle (on the x-axis) is $p - 0 = p$, and the height of the triangle (parallel to the y-axis) is $22 - 0 = 22$. Since the area is 22^2, it follows that $\frac{1}{2}p(22) = 22^2$, so $\frac{1}{2}p = 22$ and $p = 44$.

14. **Answer** 013
There are $\binom{5}{3} = C_3^5 = 10$ ways of choosing three numbers out of five, and of these choices only three are all consecutive: $1, 2, 3$ and $2, 3, 4$ and $3, 4, 5$. The probability of choosing three consecutive integers is therefore $\frac{3}{10} = \frac{p}{q}$, so $p + q = 3 + 10 = 13$.

15. **Answer** 023
Use an inductive approach. If only one pair of socks is needed it is sufficient to choose five socks. If four socks are chosen it might happen that there is one sock of each colour. If two pairs of socks are needed is sufficient to choose seven socks: among any set of seven socks there must be one pair of socks; if that pair is removed there must still be a pair of socks of the same colour among the remaining five socks already shown. On the other side there can be three green, one black, one red and one blue sock among six socks — therefore only one pair. Seven socks is there the smallest number of socks needed to ensure two pairs of socks of the same colour.
With a similar reasoning it can be shown that nine socks is sufficient to be sure that there will be three pairs of socks of the same colour; and in general, $2p + 3$ for p pairs of socks. This formula is easy to prove by mathematical induction. Hence, 23 socks are needed to be sure that there will be 10 pairs of socks of the same colour.

Alternative solution: Suppose n socks are chosen, among which there are an odd numbers of socks for k of the four colours. The number of pairs of socks is then $\frac{1}{2}(n - k)$. If n is odd, then $k = 1$ or 3, while if k is even, then $k = 0$ or 2 or 4. The least number of pairs is therefore $\frac{1}{2}(n - 3)$ if n is odd and $\frac{1}{2}(n - 4)$ if n is even. To guarantee 10 pairs, we require $\frac{1}{2}(n - 3) \geq 10$, that is, $n \geq 23$ if n is odd, and $\frac{1}{2}(n - 4) \geq 10$, that is, $n \geq 24$ if n is even. Thus, ten pairs are guaranteed if at least 23 socks are chosen.

16. **Answer** 006
To be divisible by 12, the number must be divisible by both 3 and 4, so first the sum of the digits must be divisible by 3. Of the digits $1, 2, 3, 4, 5$ the only choice of four digits with a sum divisible by 3 is $1, 2, 4, 5$. To be divisible by 4 a number formed by the last two digits must be divisible by 4, so the only possibilities for the digits of such numbers

are $12, 24, 52$. With each of these three choices the remaining two digits can be arranged in two ways in the first two places, giving a total of $3 \times 2 = 6$ choices for the required number.

17. **Answer** 010

The hikers share 10 litres equally between them, so each receives $\frac{10}{3}$ litres. Connor gives Henco $4 - \frac{10}{3} = \frac{2}{3}$ litres, and Sibu gives Henco $6 - \frac{10}{3} = \frac{8}{3}$. Since the water they donate is in the ratio $\frac{2}{3} : \frac{8}{3} = 1 : 4$, the R50 payment should be divided in the same ratio, that is, R10 to Connor and R40 to Sibu.

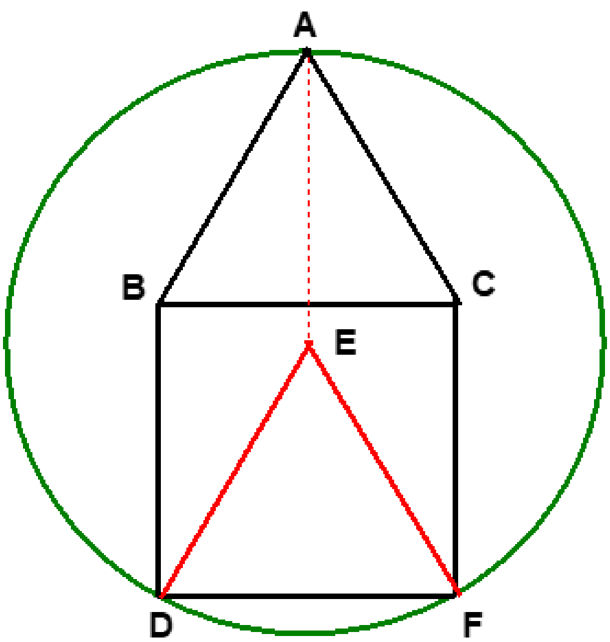
18. **Answer** 768

For the eight toppings, there are $2^8 = 256$ different selections, since each topping is either included or excluded. (This covers the possibility of no toppings at all.) Each selection of toppings can be combined with the three selections for the number of patties to give a total of $256 \times 3 = 768$ different hamburgers.

19. **Answer** 064

The sides of the square and triangle are all of length $\sqrt{64} = 8$, so the height of the triangle is $4\sqrt{3}$. Since the distance from the top vertex of the triangle to the centre of the circle is equal to the radius r, it follows that the distance from the base of the square to the centre is $8 + 4\sqrt{3} - r$. By Pythagoras' theorem it follows that $r^2 = 4^2 + (8 + 4\sqrt{3} - r)^2$, which simplifies to $r = 8$, so $r^2 = 64$.

Alternative solution: The square has all its sides equal to 8 (hence the equilateral triangle). Drop the equilateral triangle vertically downwards so that BC coincides with DF. Clearly AB and ED are equal and parallel so that ABDE is a parallelogram. However, AB = BD, so ABDE is a rhombus. And, EA = ED = EF. But then E is the centre of the circle and the radius of the circle is ED = AB = 8. The square of the radius is 64.

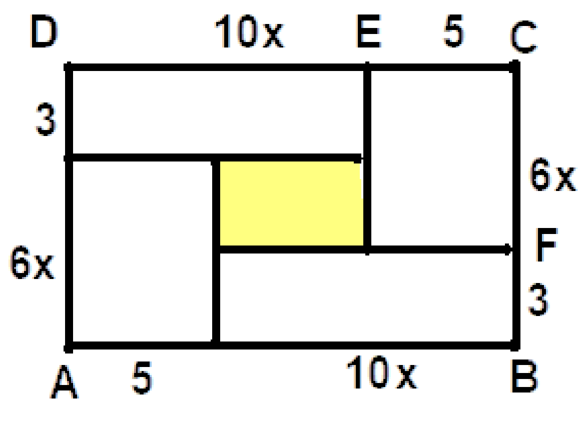


20. **Answer** 004
By long division it can be shown that $\frac{1}{7} = 0.\dot{1}42857\dot{7}\ldots$, where the block of six digits repeats (or recurs) indefinitely. Since 50 leaves remainder 2 when divided by 6, it follows that the 50-th digit after the decimal is the same as the second digit, which is 4.

21. **Answer** 017
If the two vertices at the base of the front face of the cube in the diagram are denoted R and S, then by using Pythagoras' theorem twice it can be seen that $PQ^2 = PR^2 + RQ^2 = PR^2 + (RS^2 + SQ^2) = 3PR^2$. Therefore $3PR^2 = (\sqrt[3]{17}\sqrt{3})^2 = 3 \times 17^{2/3}$, so $PR^2 = 17^{2/3}$, and the volume of the cube, which is PR^3, is equal to 17.

22. **Answer** 135
Since rectangles P and Q have the same area, suppose P is $3 \times 5k$ and Q is $5 \times 3k$, where k is unknown. Then the rectangle in the centre has dimensions $3(k-1) \times 5(k-1)$, showing that $k > 1$. Since the area of the centre rectangle is half the area of the others, it follows that $15(k-1)^2 = \frac{1}{2}(15k)$, which simplifies to $k^2 - \frac{5}{2}k + 1 = 0$ or $(k-2)(k-\frac{1}{2}) = 0$. Since $k > 1$, it follows that $k = 2$, so the area of the full rectangle is $(4 + \frac{1}{2})(15 \times 2) = 135$.

Alternative solution: Let AB $= 10x$ to avoid fractions. All the white rectangles have the same area, $3(10x) = 30x$, we can fill in the sides as indicated in the sketch below. The shaded area is a half of $30x$ which is $15x$. Now we can find the area of the large rectangle:
$(10x + 5)(6x + 3) = 4(30x) + 15x$
$(2x + 1)(2x + 1) = 8x + x = 9x$
$4x^2 - 5x + 1 = 0$
$(x - 1)(4x - 1) = 0$
$x = 0$ or $x = \frac{1}{4}$
From the diagram, $6x > 3$, so $\frac{1}{4}$ is not a solution. Hence $x = 1$ and the area of the rectangle is $(10 + 5)(3 + 6) = 15 \times 9 = 135$.



23. **Answer** 342
Suppose the numbers on the faces of the cube are A, B, C, D, E, F, with A and B on opposite faces, and similarly with F and E, and with C and D. Then faces A and F

meet faces B, C, E, D in that order, so the sum of the products at the vertices is $(A+F)(BC+CE+ED+DB) = (A+F)(B+E)(C+D) = 2022$. Thus 2022 is expressed as the product of three factors, each greater than 1 because it is the sum of two positive integers. Now 2022 is clearly divisible by 2 and 3, and by division $2022 = 2 \times 3 \times 337$. A quick check shows that 337 is not divisible by any prime number up to 17 (that is, prime numbers less than $\sqrt{337}$), so 337 is also a prime number. This means that $2 \times 3 \times 337$ is the only factorization of 2022 into three factors, so $A+F$, $B+E$ and $C+D$ are equal to 2, 3 and 337 in some order, and the sum $A+B+C+D+E+F = 2+3+337 = 342$.

24. **Answer** 506

We bracket the given sum with alternating positive and negative signs into blocks of ten numbers, and evaluate it column by column. In each block of numbers from $10k$ to $10k+9$, the alternating sum of the units digits is $-0+1-2+3-4+5-6+7-8+9=5$, and the alternating sum of the tens digits is 0, because they are all equal. Similarly, the alternating sums of the hundreds digits and the thousands digits are also equal to zero. Thus

$$-S(10k) + S(10k+1) - \cdots - S(10k+8) + S(10k+9) = 5 \text{ for all } k \geq 0.$$

(The $-S(0)$ at the beginning does not affect the answer, since $S(0) = 0$.) It follows that $S(1) - S(2) + \cdots - S(1008) + S(1009) = 101 \times 5 = 505$, since there are 101 blocks of ten numbers, and including $-S(1010) + S(1011)$ at the end gives $505 - 2 + 3 = 506$.

Alternative solution: Let $S(n)$ = sum of the digits of the positive integer n. The sequence $S(n)$ is $1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 2, 3, 4, 5, 6, 7, 8, 9, 2, 3, 4, ...$

The sum we need to calculate has terms $\pm S(n)$ where the sign is negative whenever n is even.

Let the sequence $T(n)$ be

$1, -2, 3, -4, 5, -6, 7, -8, 9, -1, 2, -3, 4, -5, 6, -7, 8, -9, 2, -3, 4, ...$

Let $R(n)$ = the sum of the first n terms of $T(n)$. We need to calculate $R(1011)$.

The sequence $R(n)$ is $1, -1, 2, -2, 3, -3, 4, -4, 5, 5, 6, 3, ...$

The odd terms appear to be the counting numbers. That is, $< R(1), R(3), R(5), R(7), ... >$ $=< 1, 2, 3, 4, ... >$.

So let us prove that the common difference of this sequence is 1.

$R(2n+1) - R(2n-1)$

$= ((T(1) + T(2) + T(3) + L + T(2n+1)$

$- ((T(1) + T(2) + T(3) + L + T(2n-1)$

$= T(2n) + T(2n+1)$

$= S(2n+1) - S(2n)$

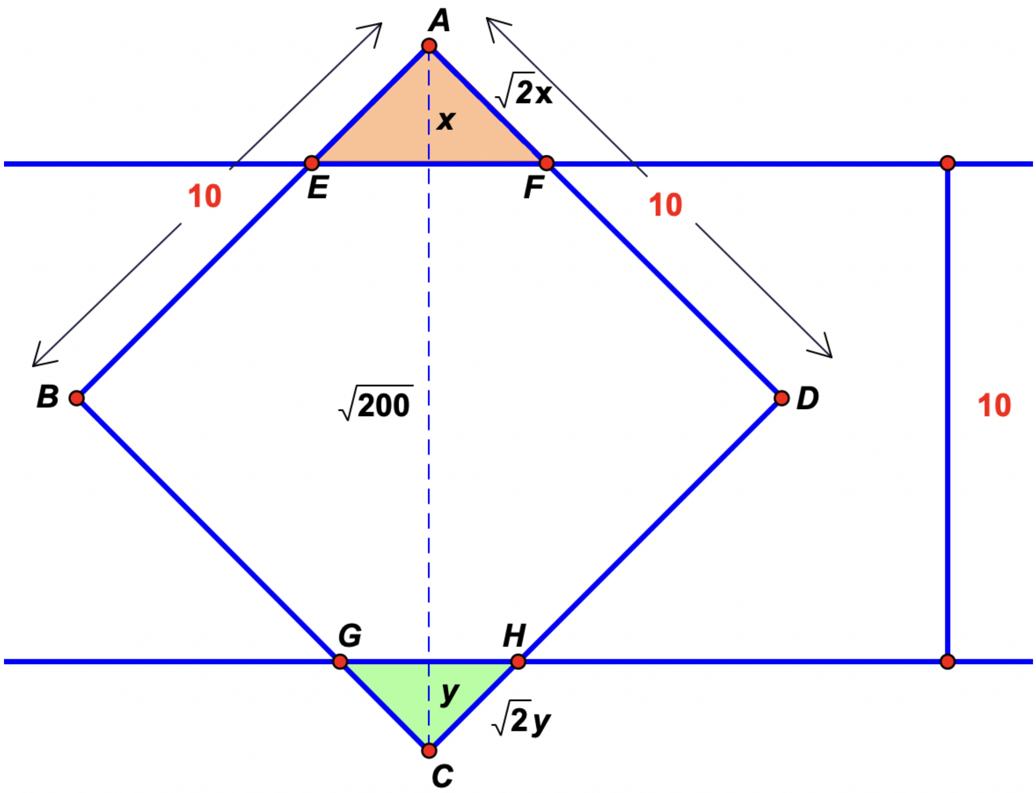
which is the difference of consecutive terms of the original sequence.

Now S increases by 1 at each natural number except where the units digit changes from 9 to 0. However, in the difference, the even number is on the right so we are interested in the difference $S(10k+1) - S(10k)$ and this is certainly 1. We have proved that the odd terms of the R sequence is the sequence of natural numbers. More precisely, $R(2n-1) = n$ for $n = 1, 2, 3, ...$.

Hence $R(1011) = R(2x506 - 1) = 506$.

25. **Answer** 020
Since the position of the square between the parallel lines, as well as the angle of tilt, is not specified, it means that we can choose the location and inclination of the square as we please. So, let two of the sides of the square be on the parallel lines. Then the shaded (degenerate) triangles have zero area but each has perimeter 10, so the sum of the perimeters is 20.

Alternative solution 1: Since the placement of the square is not specified, this implies that the sum of the perimeters remains constant no matter how the square is placed relative to the two parallel lines. For convenience and simplicity, therefore place the square as shown below so that the diagonal AC is perpendicular to the parallel lines. Let x and y be the respective heights of the isosceles triangles as shown. Then $x + y = \sqrt{200} - 10$ ... Eq. 1. The total perimeter of the two triangles $= 2(x + y) + 2\sqrt{2}(x + y)$, which after substitution of Eq. 1, simplifies to 20. This approach can be generalised.



Alternative solution 2: Let $EF = x$, $GH = y$ and $E\widehat{F}A = \theta = H\widehat{G}C$. Then the sum of the perimeters is equal to $(x+y)(\sin\theta+\cos\theta+1)$. Next, the distance between the parallel lines is equal to $BE\cos\theta + BG\sin\theta$, so $(10 - x\sin\theta)\cos\theta + (10 - y\cos\theta)\sin\theta = 10$, which simplifies to $(x + y)\sin\theta\cos\theta = 10(\sin\theta + \cos\theta - 1)$. Substituting for $(x + y)$ gives the sum of perimeters equal to

$$\frac{10}{\sin\theta\cos\theta}(\sin\theta + \cos\theta + 1)(\sin\theta + \cos\theta - 1) = \frac{10}{\sin\theta\cos\theta}((\sin\theta + \cos\theta)^2 - 1),$$

which simplifies to 20 since

$$(\sin\theta + \cos\theta)^2 - 1 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta - 1 = 2\sin\theta\cos\theta,$$

using the fact that $\sin^2\theta + \cos^2\theta = 1$.