

# A Multiple Solution Task: Another SA Mathematics Olympiad Problem

**Michael de Villiers**

**RUMEUS, University of Stellenbosch**

*profmd@mweb.co.za*

## INTRODUCTION

Challenging learners to produce multiple solutions (or proofs) for a particular problem has become an important research focus in problem solving and creativity in recent years. Research by Levav-Waynberg and Leikin (2012), for example, has indicated that a Multiple Solution Task (MST) approach in the classroom can help to promote creativity among talented learners.

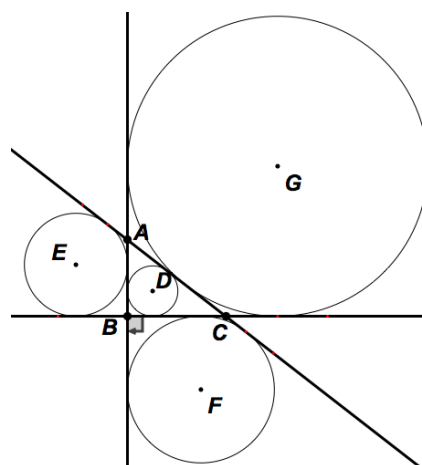
Such a task corresponds to the 'Looking Back' stage of the problem solving model of George Polya (1945) where one considers alternative solutions after the production of a solution. This process of reflection helps one to better understand the problem by viewing it from different perspectives, as well as encouraging flexibility of thought and enhancing one's repertoire of problem solving skills and approaches for future challenges. Unfortunately, in most traditional classrooms very little such opportunity for mathematically talented learners is usually created.

Although the South African Mathematics Olympiad (SAMO) doesn't explicitly ask students to arrive at multiple solutions for problems (nor does it provide an opportunity for students to write down their full solutions in the first two rounds), there are usually several problems in the first two rounds that can be solved and proved in numerous ways. In fact the potential for a problem to be solved in multiple ways is frequently used as a selection criterion, especially for harder problems.

Previously, an example from the Senior Section First Round paper of the 2016 SAMO was discussed in De Villiers (2016). In this paper I explore another MST example, this time the final question from the Senior Section Second Round paper of the 2016 SAMO<sup>2</sup>.

## THE PROBLEM

The diagram shown is formed by extending the sides of a right triangle and constructing four circles that are each tangent to all three sides. The radii of the three smaller circles with centres at D, E and F are respectively equal to 3, 4 and 21. What is the length of the radius of the circle with centre G?



<sup>2</sup> Questions and worked solutions for South African Mathematics Olympiad papers can be found at: <http://www.samf.ac.za/sa-mathematics-olympiad>  
(Note, however, that users need to register and log in to be able to view past papers and solutions.)

Underlying this problem is the rather interesting general theorem which can be stated as follows:

In a right-angled triangle, the sum of the radii of the incircle and the two excircles on the rectangular sides is equal to the radius of the excircle on the hypotenuse.

Before reading further, readers are encouraged first to dynamically explore this general theorem about right triangles at the interactive sketch at <http://dynamicmathematicslearning.com/samo2016-R2Q20.html> as well as to attempt proving the result themselves.

### THE SOLUTION

The official solution to the problem provided by SAMO goes as follows:

Let  $AB = a$  and  $BC = c$ . The points  $A, D, F$  lie on a line (the bisector of  $\widehat{BAC}$ ), so by similar triangles  $\frac{a-3}{3} = \frac{a+21}{21}$ . This gives  $a+21 = 7a-21$ , so  $6a = 42$  and  $a = 7$ . Similarly, from the points  $C, D, E$ , we have  $\frac{c+4}{4} = \frac{c-3}{3}$ , so  $4c-12 = 3c+12$  and  $c = 24$ . Now let  $T$  and  $U$  respectively be the points of tangency of the largest circle with the lines  $AC$  and  $BC$ , and let  $V$  be the foot of the perpendicular from  $D$  to  $BC$ . Quadrilateral  $GTCU$  is cyclic, because the right angles at  $T$  and  $U$  show that opposite angles are supplementary. Therefore  $\widehat{TGU} = \widehat{TCB}$  (external angle of a cyclic quadrilateral). Also  $GC$  is the bisector of  $\widehat{TGU}$ , just as  $DC$  is the bisector of  $\widehat{TCB}$ , so triangles  $GCU$  and  $CDV$  are similar. If  $g$  denotes the radius of the largest circle, then  $GU = g$  and  $UC = g - c = g - 24$ . Therefore, again by similar triangles,  $\frac{g}{g-24} = \frac{c-3}{3} = \frac{21}{3} = 7$ . Therefore  $g = 7g - 168$ , so  $6g = 168$  and  $g = 28$ .

The problem can also quite easily be solved from more general theorems that are perhaps not that well known. For example, in general, for any triangle, the reciprocal of the radius of the incircle equals the sum of the reciprocals of the radii of the 3 excircles<sup>3</sup>, from which one can easily deduce the result. Specifically, for right triangles, the sum of the radii of the three smaller circles equals the radius of the escribed circle on the hypotenuse (e.g. specifically for the given problem  $3 + 4 + 21 = 28$ , and this general result can be deduced directly from the previous theorem). Another general theorem for right triangles from which the problem can easily be solved is that, in general, the product of the radii of the incircle and the excircle on the hypotenuse equals the product of the radii of the excircles on the rectangular sides<sup>4</sup> (e.g. specifically for the given problem  $3 \times 28 = 4 \times 21$ ). Lastly, it is also possible to solve the problem by setting up coordinate axes with origin at  $B$ .

### MULTIPLE PROOFS FOR THE RIGHT TRIANGLE CENTRE-RADII THEOREM

The purpose of this paper is to explore some different proofs of the earlier stated theorem, namely that in a right-angled triangle the sum of the radii of the incircle and the two excircles on the rectangular sides is equal to the radius of the excircle on the hypotenuse. I first learnt of this very interesting result from Charles Koppelman of the Department of Mathematics at Kennesaw State University in 2006 while I was a visiting professor there and collaborating with him on setting the Kennesaw State University Annual High School Mathematics Competition. He proposed this theorem for us to consider as a problem for the Final Round of the competition (which required the writing of full proofs).

<sup>3</sup> The proof of this theorem for triangles is left as a challenge to the reader, though it features in many books and materials on problem-solving.

<sup>4</sup> The proof of this theorem for right triangles is also left as a challenge to the reader.

On reflection, however, I felt the problem would likely be too easy for experienced problem-solving students who might know the two theorems mentioned earlier, namely, (i) that the reciprocal of the radius of the incircle of a triangle equals the sum of the reciprocals of the radii of the three excircles, and (ii) that for right triangles, the product of the radii of the incircle and the excircle on the hypotenuse equals the product of the radii of the excircles on the rectangular sides. From these two theorems the result could then be derived as follows:

**PROOF 1**

With reference to Figure 1, we have from the second theorem for right triangles that  $r_D \times r_G = r_E \times r_F \Rightarrow r_D = \frac{r_E \times r_F}{r_G}$ .

Substituting the latter into the first theorem for triangles gives the following:  $\frac{1}{r_E} + \frac{1}{r_F} + \frac{1}{r_G} = \frac{r_G}{r_E \times r_F}$ . After multiplying through by  $r_E \times r_F$ , simplifying and rearranging, we obtain the desired result:  $r_E + r_F + r_D = r_G$ .

However, the result can also be derived and proved from first principles as shown in the following proof produced by Koppelman.

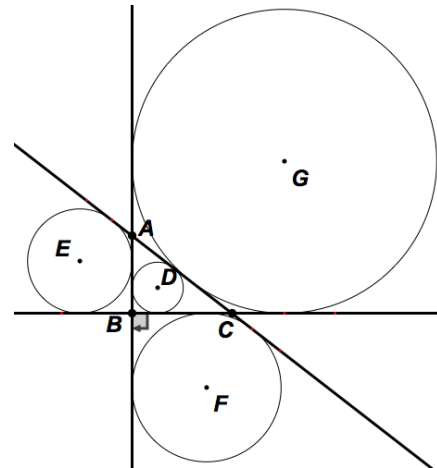


FIGURE 1

**PROOF 2**

For the right triangle ABC, let  $AB = a$ ,  $BC = b$  and  $AC = c$ . First consider the inner circle as shown in Figure 2. Noting that the tangents to a circle from an external point are equal in length, label the diagram as indicated. We now have  $(a - x) + (b - x) = c$ , from which it follows that  $x = \frac{a + b - c}{2}$ , i.e.  $r_D = \frac{a + b - c}{2}$ .

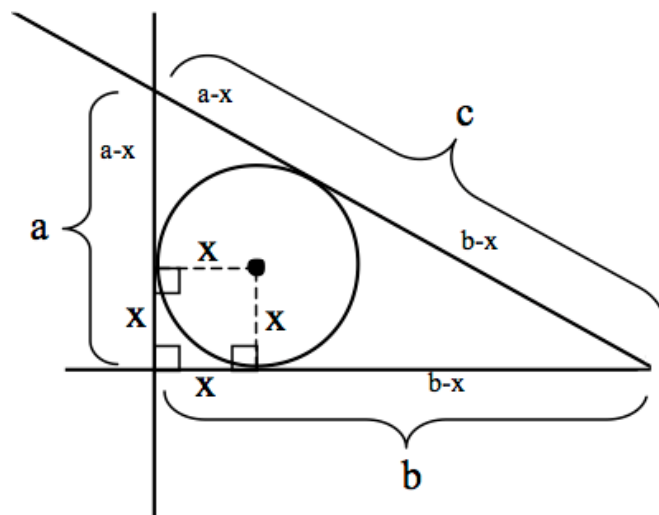


FIGURE 2

Next consider the largest circle and label the diagram as shown in Figure 3. We now have  $(x-a)+(x-b)=c$  from which it follows that  $x = \frac{a+b+c}{2}$ , i.e.  $r_G = \frac{a+b+c}{2}$ .

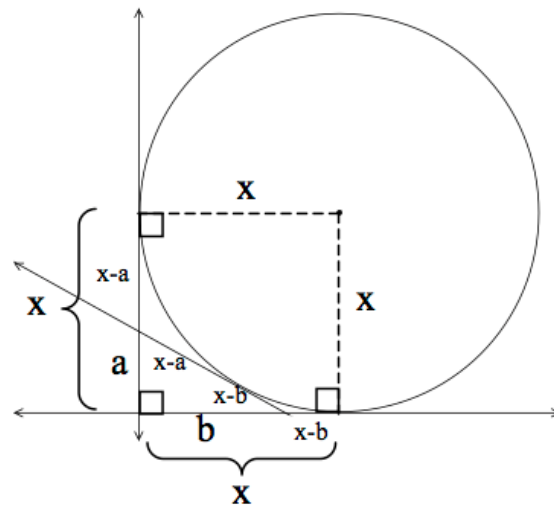


FIGURE 3

Next consider either of the remaining two circles and label the diagram in Figure 4 as indicated. We now have  $c+a-x=b+x$  from which it follows that  $x = \frac{c+a-b}{2}$ , i.e.  $r_E = \frac{c+a-b}{2}$ .

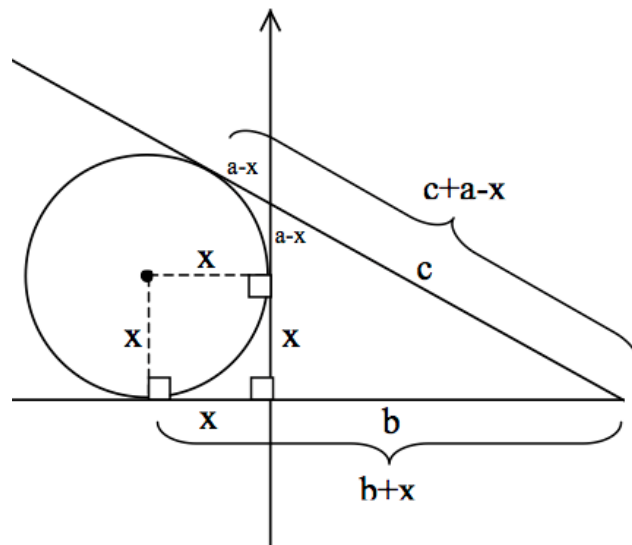


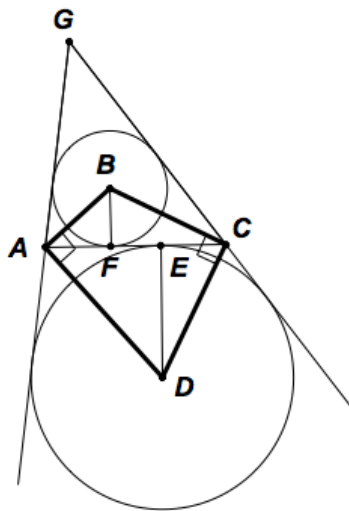
FIGURE 4

Finally, using the same argument on the remaining circle with  $a$  and  $b$  interchanged gives  $r_F = \frac{c+b-a}{2}$ . Adding the lengths of the radii of the three smaller circles,  $\frac{a+b-c}{2} + \frac{c+a-b}{2} + \frac{c+b-a}{2}$  simplifies to  $\frac{a+b+c}{2}$  which is the same as the radius of the largest circle.

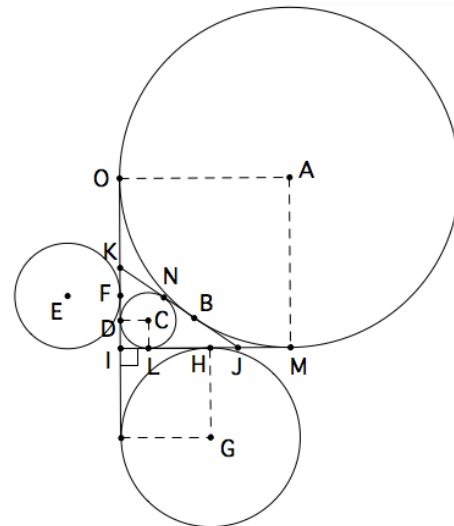
**PROOF 3**

One often finds in problem solving that problems are connected, and that one can sometimes use theorems from an old, solved problem in tackling new problems. The next proof by myself uses the following useful theorem, which came up in relation to a seemingly unrelated problem in the 2006 SAMO, involving a quadrilateral ABCD with a pair of opposite right angles at A and C, as discussed and proved in De Villiers (2007).

Theorem: Given any triangle GAC as shown in Figure 5, then if F and E are the respective tangent points of the triangle's incircle and excircle to AC, then  $AE = CF$  (and  $CE = AF$ ).



**FIGURE 5**



**FIGURE 6**

Consider Figure 6. Let the radii of circles A, C, E and G respectively be  $r_3, r, r_1,$  and  $r_2$ . Since  $\hat{K}IJ$  is a right angle, firstly note the three squares formed by the three quadrilaterals with opposite vertices respectively labelled D and L, I and G, and A and I. Using the theorem above we have  $IL = r = JH$ . Thus  $IJ = r + r_2$ . Similarly,  $IK = r + r_1$ . Since tangents from a common point to a circle are equal in length,  $KD = KN = r_1$  and  $JN = JL = r_2$ , and it follows that  $KJ = r_1 + r_2$ . But  $OK = r_3 - r - r_1 = KB$  and  $MJ = r_3 - r - r_2 = BJ$ ; so therefore  $KJ = KB + BJ = 2r_3 - 2r - r_1 - r_2 = r_1 + r_2$  (from before). Solving this for  $r_3$  gives the desired result:  $r_3 = r + r_1 + r_2$ .

**CONCLUDING COMMENTS**

In the spirit of this paper, readers are encouraged and invited to submit interesting alternative solutions for this interesting, little-known theorem or any other SAMO questions to *Learning & Teaching Mathematics*, as it is likely that the official solutions are often not the only ones, nor necessarily the most elegant ones.

**REFERENCES**

De Villiers, M. (2007). Further reflection on a SA Mathematics Olympiad problem, *Learning & Teaching Mathematics*, 4, 25-27.

De Villiers, M. (2016). A multiple solution task: A South African Mathematics Olympiad problem. *Learning & Teaching Mathematics*, 20, 18-20.

Levav-Waynberg, A. & Leikin, R. (2012). The role of multiple solution tasks in developing knowledge and creativity in geometry. *Journal of Mathematical Behavior*, 31, 73-90. doi:10.1016/j.jmathb.2011.11.001

Polya, G. (1945). *How to solve it*. Princeton: Princeton University Press.