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A dual, and generalisations, of a Sharp result

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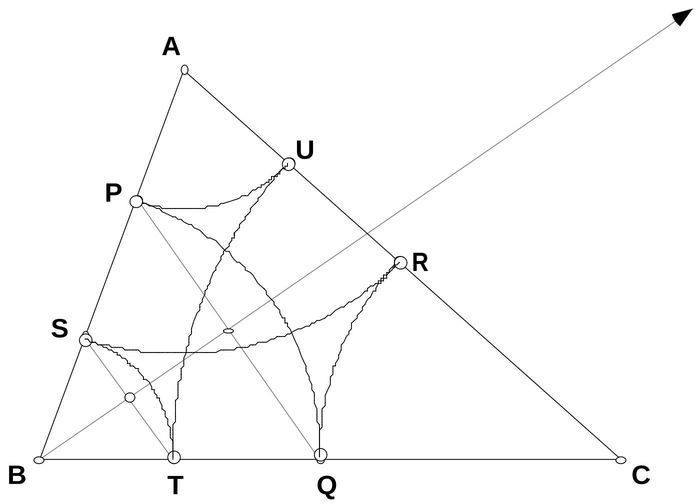
http://mzone.mweb.co.za/residents/profmd/homepage2.html

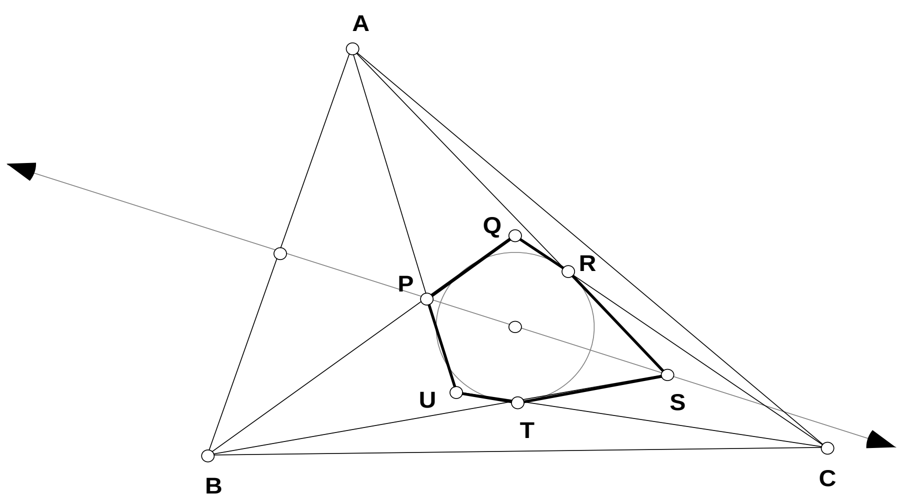
In the *Mathematical Digest*, No. 117, Oct 1999, solutions to the following Sharp problem were given by several learners on p. 17:



In triangle ABC point P lies on AB. Six circular arcs are draw:
With centre B and radius BP, cutting BC in Q,
With centre C and radius CQ, cutting CA in R,
with centre A and radius AR, cutting AB in S,
with centre B and radius BS, cutting BC in T,
with centre C and radius CT, cutting CA in U,
with centre A and radius AU.
Prove that this last arc cuts AB in P.

However, more can be said about this configuration, namely, that the points P, Q, R, S, T and U all lie on a circle, i.e. the (crossed) hexagon PQRSTU is cylic. This is easy to prove as follows: Since PQB is an isosceles triangle with BP = BQ, the perpendicular bisector of PQ coincides with the angle bisector of angle B. Similarly, the perpendicular bisectors of QR, RS, etc. coincide with the angle bisectors of angles C, A, etc. Since the angle bisectors of any triangle are concurrent, it follows that the perpendicular bisectors of the sides of PQRSTU are also concurrent, and therefore it is cyclic. (Since a perpendicular bisector is the locus of all points *equidistant* from two endpoints of a segment and the perpendicular bisectors of all six sides are concurrent, this point of concurrency (the incentre) is *equidistant* from all six vertices).

The result is also true if P lies on AB extended either way and one works with directed line segments. Furthermore, as pointed out in De Villiers (1996: 197-198; 202) there is an interesting dual to this result involving *angles* rather than *sides*. Whereas with the first result we start with an arbitrary point P on AB, we now start with an arbitrary ray dividing angle A. For example, construct any angle divider $\overrightarrow{AP}$ of $\angle A$ of a triangle ABC, angle divider $\overrightarrow{BP}$ of $\angle B$ so that $\angle PBA = \angle PAB$, angle divider $\overrightarrow{CQ}$ of $\angle C$ so that $\angle QCB = \angle PBC$ and $Q \in \overrightarrow{BP}$, angle divider $\overrightarrow{AR}$ of $\angle A$ so that $\angle RAC = \angle QCA$ and $R \in \overrightarrow{CQ}$, angle divider $\overrightarrow{BS}$ of $\angle B$ so that $\angle SBA = \angle RAB$ and $S \in \overrightarrow{AR}$ and angle divider $\overrightarrow{CT}$ of $\angle C$ so that $\angle TCB = \angle SBC$ and $T \in \overrightarrow{BS}$. If U is the intersection of $\overrightarrow{CT}$ and $\overrightarrow{AP}$, then $\angle UCA = \angle UAC$ and PQRSTU is a circum hexagon (a hexagon circumscribed around a circle - see below).



The proof is similar to the original and is left as an exercise to the reader. As pointed out in De Villiers (1996: 58-61; 196) both results can be respectively generalised to circum

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and cyclic polygons. An even further generalisation which does not retain the cyclic or circum property is also discussed.

Reference

De Villiers, M. (1996). **Some Adventures in Euclidean Geometry**. Univ. Durban-Westville. (R22 for AMESA members; R26 for non-members).

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The Sharp problem mentioned above was poetically described as follows by David Gale in a recent issue of the *Mathematical Intelligencer*:

In a triangle called *ABC*
Pick a point on *AB*, call it *P*.
Pick a *Q* on *BC*,
Where *BQ* is *BP*.
Ah the joys of pure geo-me-tree!

On *CA* pick an *R*, oh please do,
Where *CR* is exactly *CQ*,
And now pick an *S*
On *AB*, more or less,
So that "*AS* is *AR*" is true.

On *BC* the next letter is *T*,
Where *BT* is *BS*, don't you see.
On *CA* pick a *U*,
And you'll know what to do,
Next what's this? We've arrived back at *P*!

Now some proofs were soon found close at hand,
But it did'nt turn out quite as planned,
For though not very large
(They would fit in the margin)
regrettably, none of them scanned.