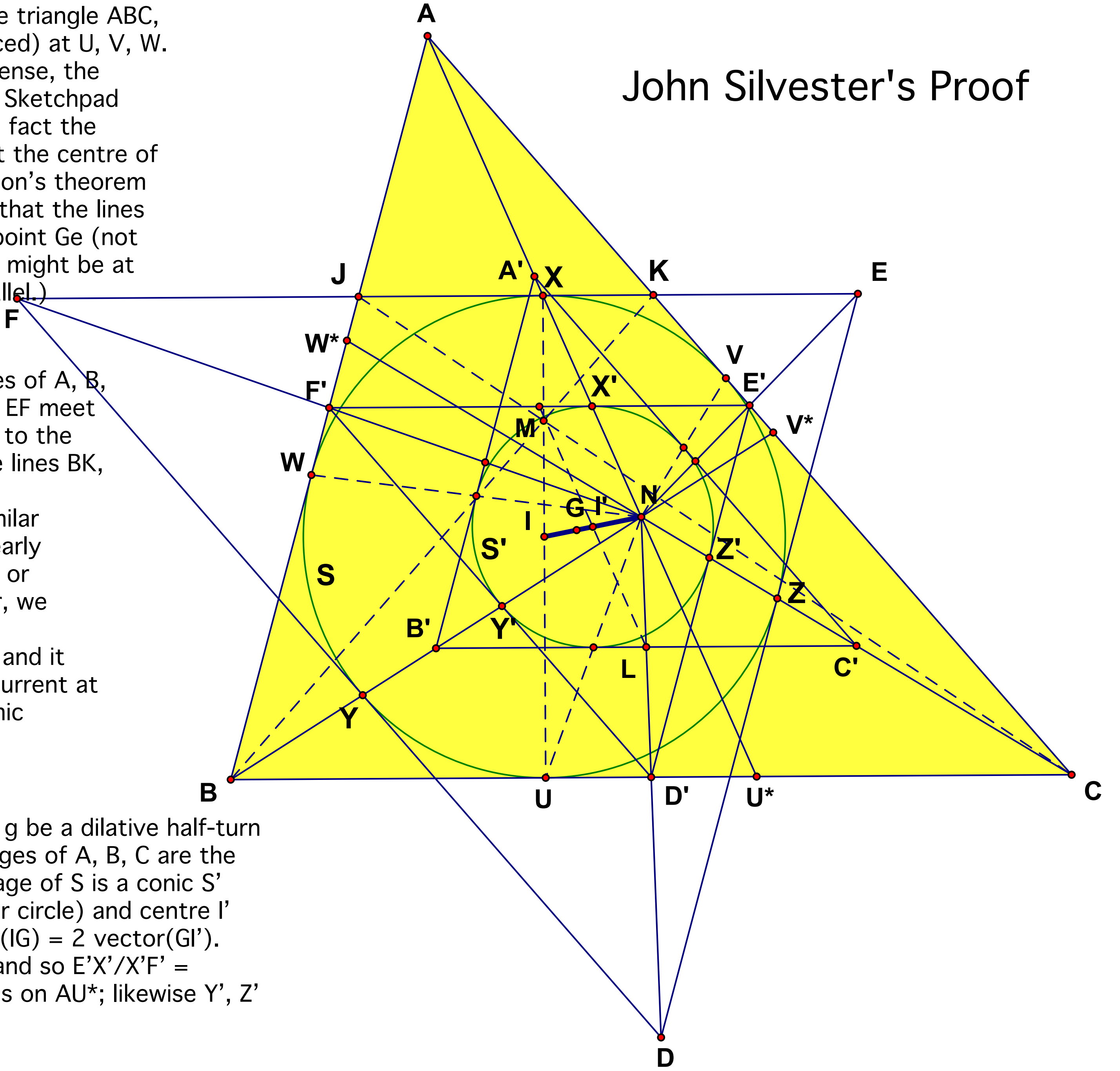
John Silvester's Proof

Let S be a proper central conic inscribed in the triangle ABC, touching its sides BC, CA, AB (possibly produced) at U, V, W. Think of this as a generalised incircle. In this sense, the escribed circles are generalised incircles! (The Sketchpad diagram, page 1, shows the case where S is in fact the incircle; page 2 shows the escribed circle.) Let the centre of S be I (generalised incentre). Applying Brianchon’s theorem to the degenerate hexagon BUCVAW, we see that the lines AU, BV, CW (not shown) are concurrent at a point Ge (not shown; it is the generalised Gergonne point. It might be at infinity: the lines AU, BV, CW might be all parallel.)

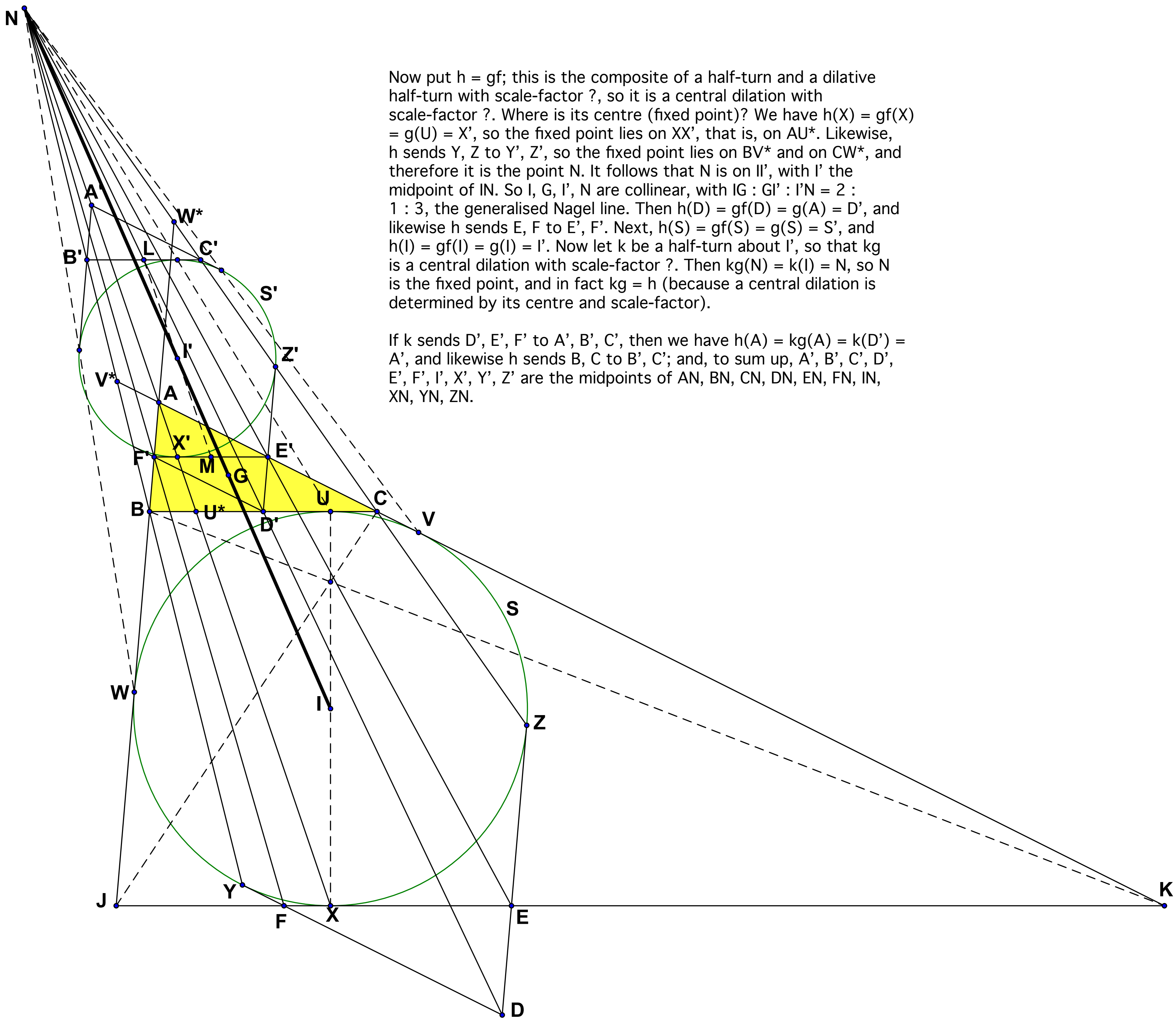


Apply a half-turn f, centre I, and let the images of A, B, C, U, V, W be D, E, F, X, Y, Z respectively. Let EF meet AB, AC at J, K. Applying Brianchon’s theorem to the degenerate hexagon BUCKXJ, we see that the lines BK, UX, CJ are concurrent (or all parallel), as a consequence of which BU/UC = KX/XJ (by similar triangles). But, if AX meets BC at U*, then clearly KX/XJ = CU*/U*B, whence BU/UC = CU*/U*B, or vector(BU) = vector(U*C). In a similar manner, we obtain V*, W* on CA, AB, with vector(CV) = vector(V*A) and vector(AW) = vector(W*B), and it follows that the lines AU*, BV*, CW* are concurrent at a point N (generalised Nagel point, the isotomic conjugate of Ge.)

Let G be the centroid of triangle ABC, and let g be a dilative half-turn about G, with scale factor ?. Under g, the images of A, B, C are the midpoints D’, E’, F’ of BC, CA, AB; and the image of S is a conic S’ inscribed in triangle D’E’F’ (generalised Spieker circle) and centre I’ (generalised Spieker centre, on IG with vector(IG) = 2 vector(GI’). Under g, the images of U, V, W are X’, Y’, Z’, and so E’X’/X’F’ = BU/UC= CU*/U*B, from which we see that X’ is on AU*; likewise Y’, Z’ are on BV*, CW*.

Now put h = gf; this is the composite of a half-turn and a dilative half-turn with scale-factor ?, so it is a central dilation with scale-factor ?. Where is its centre (fixed point)? We have h(X) = gf(X) = g(U) = X’, so the fixed point lies on XX’, that is, on AU*. Likewise, h sends Y, Z to Y’, Z’, so the fixed point lies on BV* and on CW*, and therefore it is the point N. It follows that N is on II’, with I’ the midpoint of IN. So I, G, I’, N are collinear, with IG : GI’ : I’N = 2 : 1 : 3, the generalised Nagel line. Then h(D) = gf(D) = g(A) = D’, and likewise h sends E, F to E’, F’. Next, h(S) = gf(S) = g(S) = S’, and h(I) = gf(I) = g(I) = I’. Now let k be a half-turn about I’, so that kg is a central dilation with scale-factor ?. Then kg(N) = k(I) = N, so N is the fixed point, and in fact kg = h (because a central dilation is determined by its centre and scale-factor).

If k sends D’, E’, F’ to A’, B’, C’, then we have h(A) = kg(A) = k(D’) = A’, and likewise h sends B, C to B’, C’; and, to sum up, A’, B’, C’, D’, E’, F’, I’, X’, Y’, Z’ are the midpoints of AN, BN, CN, DN, EN, FN, IN, XN, YN, ZN.



Locating I’, the generalised Spieker centre: h(D’) = midpoint of D’N, which we’ll call L. Since D’ is the midpoint of BC, L must be the midpoint of B’C’. Then k sends B’, C’ to E’, F’, so k(L) is the midpoint of E’F’, which we’ll call M. Finally, I’ is the midpoint of LM. So to locate I’, we only need the three midpoints D’, E’, F’ and the generalised Nagel point N: take L as midpoint of D’N and M as midpoint of E’F’, and then I’ is the midpoint of LM. If you want to locate I as well, then extend NI’ and mark I with vector(NI’) = vector(I’I). I have done this construction on the Sketchpad diagram, page 3, which allows you to drag the generalised Gergonne point Ge, and shows the Steiner circumellipse. Here S is a hyperbola, but if you bring Ge inside the Steiner circumellipse, S becomes an ellipse.