Forum Geometricorum
Volume 12 (2012) 63–77.

FORUM GEOM
ISSN 1534-1178

Similar Metric Characterizations of Tangential and Extangential Quadrilaterals

Martin Josefsson

Abstract. We prove five necessary and sufficient conditions for a convex quadrilateral to have an excircle and compare them to similar conditions for a quadrilateral to have an incircle.

1. Introduction

There are a lot of more or less well known characterizations of tangential quadrilaterals,[1] that is, convex quadrilaterals with an incircle. This circle is tangent at the inside of the quadrilateral to all four sides. Many of these necessary and sufficient conditions were either proved or reviewed in [8]. In this paper we shall see that there are a few very similar looking characterizations for a convex quadrilateral to have an *excircle*. This is a circle that is tangent at the outside of the quadrilateral to the extensions of all four sides. Such a quadrilateral is called an *extangential quadrilateral* in [13, p.44],[2] see Figure 1.

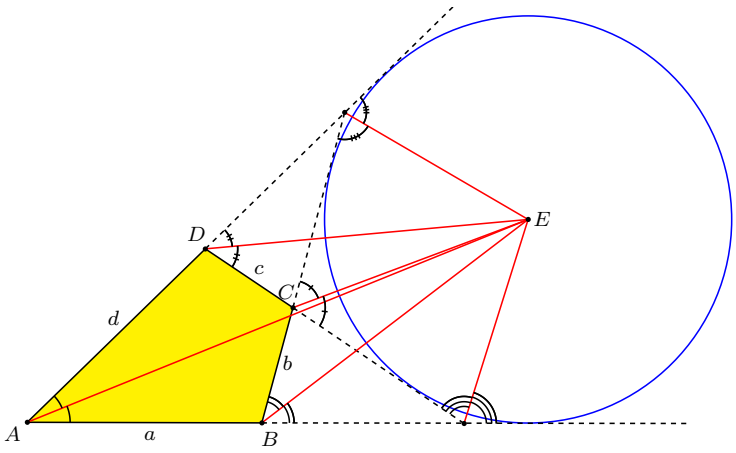

Figure 1. An extangential quadrilateral and its excircle

We start by reviewing and commenting on the known characterizations of extangential quadrilaterals and the similar ones for tangential quadrilaterals. It is well known that a convex quadrilateral is tangential if and only if the four internal angle

Publication Date: April 4, 2012. Communicating Editor: Paul Yiu.
[1]Another common name for these is circumscribed quadrilateral.
[2]Alexander Bogomolny calls them exscriptible quadrilateral at [2].

bisectors to the vertex angles are concurrent. Their common point is the incenter, that is, the center of the incircle. A convex quadrilateral is extangential if and only if six angles bisectors are concurrent, which are the internal angle bisectors at two opposite vertex angles, the external angle bisectors at the other two vertex angles, and the external angle bisectors at the angles formed where the extensions of opposite sides intersect. Their common point is the excenter (E in Figure 1).

The most well known and useful characterization of tangential quadrilaterals is the Pitot theorem, that a convex quadrilateral with sides a, b, c, d has an incircle if and only if opposite sides have equal sums,

$$a + c = b + d.$$

For the existence of an excircle, the similar characterization states that the adjacent sides shall have equal sums. This is possible in two different ways. There can only be one excircle to a quadrilateral, and the characterization depends on which pair of opposite vertices the excircle is outside of. It is easy to realize that it must be outside the vertex (of the two considered) with the biggest angle.[3] A convex quadrilateral $ABCD$ has an excircle outside one of the vertices A or C if and only if

$$a + b = c + d \tag{1}$$

according to [2] and [10, p.69]. This was proved by the Swiss mathematician Jakob Steiner (1796–1863) in 1846 (see [3, p.318]). By symmetry ($b \leftrightarrow d$), there is an excircle outside one of the vertices B or D if and only if

$$a + d = b + c. \tag{2}$$

From (1) and (2), we have that a convex quadrilateral with sides a, b, c, d has an excircle if and only if

$$|a - c| = |b - d|$$

which resembles the Pitot theorem. There is however one exception to these characterizations. The existence of an excircle is dependent on the fact that the extensions of opposite sides in the quadrilateral intersect, otherwise the circle can never be tangent to all four extensions. Therefore there is no excircle to either of a trapezoid, a parallelogram, a rhombus, a rectangle or a square even though (1) or (2) is satisfied in many of them, since they have at least one pair of opposite parallel sides.[4]

In [8, p.66] we reviewed two characterizations of tangential quadrilaterals regarding the extensions of the four sides. Let us take another look at them here. If $ABCD$ is a convex quadrilateral where opposite sides AB and CD intersect at E, and the sides AD and BC intersect at F (see Figure 2), then $ABCD$ is a tangential quadrilateral if and only if either of the following conditions holds:

$$AE + CF = AF + CE, \tag{3}$$

$$BE + BF = DE + DF. \tag{4}$$

[3] Otherwise the circle can never be tangent to all four extensions.

[4] The last four of these quadrilaterals can be considered to be extangential quadrilaterals with infinite exradius, see Theorem 8.

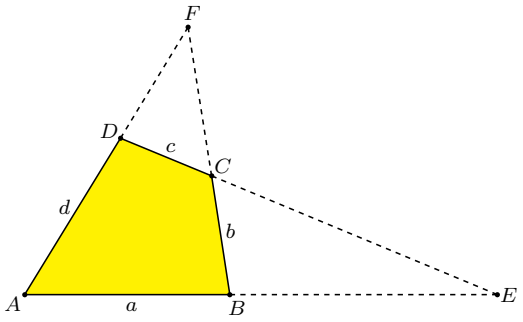

Figure 2. The extensions of the sides

The history of these conditions are discussed in [14] together with the corresponding conditions for extangential quadrilaterals. In our notations, $ABCD$ has an excircle outside one of the vertices A or C if and only if either of the following conditions holds:

$$AE + CE = AF + CF, \tag{5}$$

$$BE + DE = BF + DF. \tag{6}$$

These conditions were stated somewhat differently in [14] with other notations. Also, there it was not stated that the excircle can be outside A instead of C, but that is simply a matter of making the change $A \leftrightarrow C$ in (5) to see that the condition is unchanged. How about an excircle outside of B or D? By making the changes $A \leftrightarrow D$ and $B \leftrightarrow C$ (to preserve that AB and CD intersect at E) we find that the conditions (5) and (6) are still the same. According to [14], conditions (3) and (5) were proved by Jakob Steiner in 1846. In 1973, Howard Grossman (see [5]) contributed with the two additional conditions (4) and (6).

From a different point of view, (3) and (5) can be considered to be necessary and sufficient conditions for when a *concave* quadrilateral $AECF$ has an "incircle" (a circle tangent to two adjacent sides and the extensions of the other two) or an excircle respectively. Then (4) and (6) are necessary and sufficient conditions for a *complex* quadrilateral $BEDF$ to have an excircle.[5]

Another related theorem is due to the Australian mathematician M. L. Urquhart (1902–1966). He considered it to be "the most elementary theorem of Euclidean geometry". It was originally stated using only four intersecting lines. We restate it in the framework of a convex quadrilateral $ABCD$, where opposite sides intersect at E and F, see Figure 2. Urquhart's theorem states that if $AB+BC = AD+DC$, then $AE + EC = AF + FC$. In 1976 Dan Pedoe wrote about this theorem (see [12]), where he concluded that the proof by purely geometrical methods is not elementary and that he had been trying to find such a proof that did not involve a circle (the excircle to the quadrilateral). Later that year, Dan Sokolowsky took up

[5]Equations (4) and (6) can then be merged into one as $|BE - DF| = |BF - DE|$.

that challenge and gave an elementary "no-circle" proof in [15]. In 2006, Mowaffaq Hajja gave a simple trigonometric proof (see [6]) that the two equations in Urquhart's theorem are equivalent. According to (1) and (5), they are both characterizations of an extangential quadrilateral $ABCD$.

2. Characterizations with subtriangle circumradii

In [9, pp.23–24] we proved that if the diagonals in a convex quadrilateral $ABCD$ intersect at P, then it has an incircle if and only if

$$R_1 + R_3 = R_2 + R_4$$

where R_1, R_2, R_3 and R_4 are the circumradii in the triangles ABP, BCP, CDP and DAP respectively, see Figure 3.

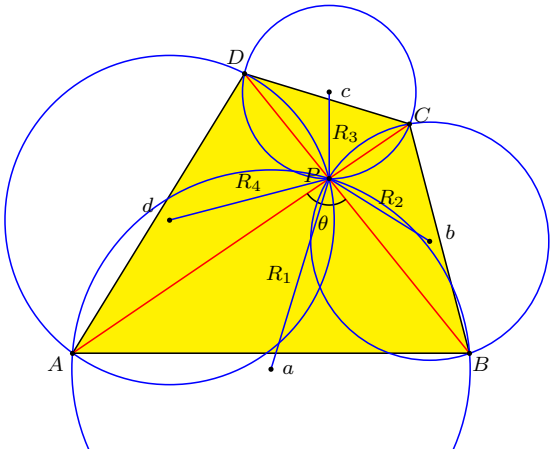

Figure 3. The subtriangle circumcircles

There are the following similar conditions for a quadrilateral to have an excircle.

Theorem 1. *Let R_1, R_2, R_3, R_4 be the circumradii in the triangles ABP, BCP, CDP, DAP respectively in a convex quadrilateral $ABCD$ where the diagonals intersect at P. It has an excircle outside one of the vertices A or C if and only if*

$$R_1 + R_2 = R_3 + R_4$$

and an excircle outside one of the vertices B or D if and only if

$$R_1 + R_4 = R_2 + R_3.$$

Proof. According to the extended law of sines, the sides satisfies $a = 2R_1 \sin\theta$, $b = 2R_2 \sin\theta$, $c = 2R_3 \sin\theta$ and $d = 2R_4 \sin\theta$, where θ is the angle between the diagonals,[6] see Figure 3. Thus

$$a + b - c - d = 2\sin\theta(R_1 + R_2 - R_3 - R_4)$$

[6]We used that $\sin(\pi - \theta) = \sin\theta$ to get two of the formulas.

and

$$a + d - b - c = 2 \sin\theta (R_1 + R_4 - R_2 - R_3).$$

From these we directly get that

$$a + b = c + d \quad \Leftrightarrow \quad R_1 + R_2 = R_3 + R_4$$

and

$$a + d = b + c \quad \Leftrightarrow \quad R_1 + R_4 = R_2 + R_3$$

since $\sin\theta \neq 0$. By (1) and (2) the conclusions follow. □

3. Characterizations concerning the diagonal parts

In [7] Larry Hoehn made a few calculations with the law of cosines to prove that in a convex quadrilateral $ABCD$ with sides a, b, c, d,

$$efgh(a+c+b+d)(a+c-b-d) = (agh+cef+beh+dfg)(agh+cef-beh-dfg)$$

where e, f, g, h are the distances from the vertices A, B, C, D respectively to the diagonal intersection (see Figure 4). Using the Pitot theorem $a + c = b + d$, we get that the quadrilateral is tangential if and only if

$$agh + cef = beh + dfg. \tag{7}$$

Now we shall prove that there are similar characterizations for the quadrilateral to have an excircle.

Theorem 2. *Let e, f, g, h be the distances from the vertices A, B, C, D respectively to the diagonal intersection in a convex quadrilateral $ABCD$ with sides a, b, c, d. It has an excircle outside one of the vertices A or C if and only if*

$$agh + beh = cef + dfg$$

and an excircle outside one of the vertices B or D if and only if

$$agh + dfg = beh + cef.$$

Proof. In [7] Hoehn proved that in a convex quadrilateral,

$$efgh\left(a^2 + c^2 - b^2 - d^2\right) = a^2g^2h^2 + c^2e^2f^2 - b^2e^2h^2 - d^2f^2g^2.$$

Now adding $efgh(-2ac + 2bd)$ to both sides, this is equivalent to

$$efgh\left((a-c)^2 - (b-d)^2\right) = (agh - cef)^2 - (beh - dfg)^2$$

which is factored as

$$efgh(a-c+b-d)(a-c-b+d) = (agh-cef+beh-dfg)(agh-cef-beh+dfg).$$

The left hand side is zero if and only if $a+b = c+d$ or $a+d = b+c$ and the right hand side is zero if and only if $agh+beh = cef+dfg$ or $agh+dfg = beh+cef$.

To show that the first equality from both sides are connected and that the second equality from both sides are also connected, we study a special case. In a kite where $a = d$ and $b = c$ and also $f = h$, the two equalities $a + b = c + d$ and $agh + beh = cef + dfg$ are satisfied, but none of the others. This proves that they

are connected. In the same way, using another kite, the other two are connected and we have that

$$a+b=c+d \quad \Leftrightarrow \quad agh+beh=cef+dfg$$

and

$$a+d=b+c \quad \Leftrightarrow \quad agh+dfg=beh+cef.$$

This completes the proof according to (1) and (2). □

Remark. The characterization (7) had been proved at least three different times before Hoehn did it. It appears as part of a proof of an inverse inradii characterization of tangential quadrilaterals in [16] and [17]. It was also proved in [11, Proposition 2 (e)]. All of the four known proofs used different notations.

4. Characterizations with subtriangle altitudes

In 2009, Nicuşor Minculete gave two different proofs (see [11]) that a convex quadrilateral $ABCD$ has an incircle if and only if the altitudes h_1, h_2, h_3, h_4 from the diagonal intersection P to the sides AB, BC, CD, DA in triangles ABP, BCP, CDP, DAP respectively satisfy

$$\frac{1}{h_1}+\frac{1}{h_3}=\frac{1}{h_2}+\frac{1}{h_4}. \tag{8}$$

This characterization of tangential quadrilaterals had been proved as early as 1995 in Russian by Vasilyev and Senderov [16]. Another Russian proof was given in 2004 by Zaslavsky [18]. To prove that (8) holds in a tangential quadrilateral (i.e. not the converse) was a problem at the 2009 mathematics Olympiad in Germany [1]. All of these but the 1995 proof used other notations.

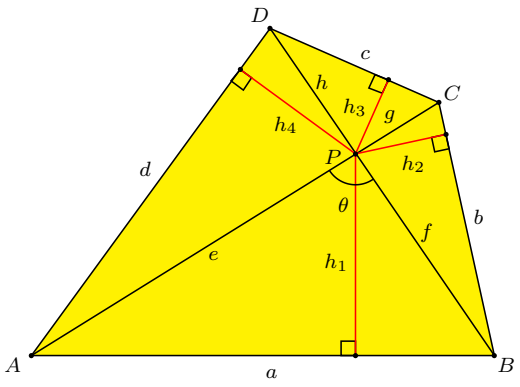

Figure 4. The subtriangle altitudes h_1, h_2, h_3 and h_4

Here we will give a short fifth proof that (8) is a necessary and sufficient condition for a convex quadrilateral to have an incircle using the characterization (7).

By expressing twice the area of ABP, BCP, CDP, DAP in two different ways, we have the equalities (see Figure 4)

$$\begin{aligned} ah_1 &= ef\sin\theta, \\ bh_2 &= fg\sin\theta, \\ ch_3 &= gh\sin\theta, \\ dh_4 &= he\sin\theta \end{aligned} \tag{9}$$

where θ is the angle between the diagonals.[7] Hence

$$\left(\frac{1}{h_1}+\frac{1}{h_3}-\frac{1}{h_2}-\frac{1}{h_4}\right)\sin\theta = \frac{a}{ef}+\frac{c}{gh}-\frac{b}{fg}-\frac{d}{he} = \frac{agh+cef-beh-dfg}{efgh}.$$

Since $\sin\theta \neq 0$, we have that

$$\frac{1}{h_1}+\frac{1}{h_3} = \frac{1}{h_2}+\frac{1}{h_4} \quad \Leftrightarrow \quad agh+cef = beh+dfg$$

which by (7) proves that (8) is a characterization of tangential quadrilaterals.

Now we prove the similar characterizations of extangential quadrilaterals.

Theorem 3. *Let h_1, h_2, h_3, h_4 be the altitudes from the diagonal intersection P to the sides AB, BC, CD, DA in the triangles ABP, BCP, CDP, DAP respectively in a convex quadrilateral $ABCD$. It has an excircle outside one of the vertices A or C if and only if*

$$\frac{1}{h_1}+\frac{1}{h_2} = \frac{1}{h_3}+\frac{1}{h_4}$$

and an excircle outside one of the vertices B or D if and only if

$$\frac{1}{h_1}+\frac{1}{h_4} = \frac{1}{h_2}+\frac{1}{h_3}.$$

Proof. The four equations (9) yields

$$\left(\frac{1}{h_1}+\frac{1}{h_2}-\frac{1}{h_3}-\frac{1}{h_4}\right)\sin\theta = \frac{a}{ef}+\frac{b}{fg}-\frac{c}{gh}-\frac{d}{he} = \frac{agh+beh-cef-dfg}{efgh}.$$

Since $\sin\theta \neq 0$, we have that

$$\frac{1}{h_1}+\frac{1}{h_2} = \frac{1}{h_3}+\frac{1}{h_4} \quad \Leftrightarrow \quad agh+beh = cef+dfg$$

which by Theorem 2 proves the first condition in the theorem. The second is proved in the same way. □

[7] Here we have used that $\sin(\pi-\theta) = \sin\theta$ in two of the equalities.

5. Iosifescu's characterization for excircles

According to [11, p.113], Marius Iosifescu proved in 1954 that a convex quadrilateral $ABCD$ has an incircle if and only if

$$\tan\frac{x}{2}\tan\frac{z}{2}=\tan\frac{y}{2}\tan\frac{w}{2}$$

where $x=\angle ABD$, $y=\angle ADB$, $z=\angle BDC$ and $w=\angle DBC$, see Figure 5. That proof was given in Romanian, but an English one was given in [8, pp.75–77].

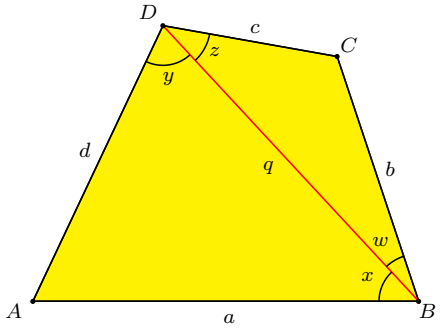

Figure 5. Angles in Iosifescu's characterization

There are similar characterizations for a quadrilateral to have an excircle, which we shall prove in the next theorem.

Theorem 4. *Let $x=\angle ABD$, $y=\angle ADB$, $z=\angle BDC$ and $w=\angle DBC$ in a convex quadrilateral $ABCD$. It has an excircle outside one of the vertices A or C if and only if*

$$\tan\frac{x}{2}\tan\frac{w}{2}=\tan\frac{y}{2}\tan\frac{z}{2}$$

and an excircle outside one of the vertices B or D if and only if

$$\tan\frac{x}{2}\tan\frac{y}{2}=\tan\frac{z}{2}\tan\frac{w}{2}.$$

Proof. In [8], Theorem 7, we proved by using the law of cosines that

$$1-\cos x=\frac{(d+a-q)(d-a+q)}{2aq},\quad 1+\cos x=\frac{(a+q+d)(a+q-d)}{2aq},$$
$$1-\cos y=\frac{(a+d-q)(a-d+q)}{2dq},\quad 1+\cos y=\frac{(d+q+a)(d+q-a)}{2dq},$$
$$1-\cos z=\frac{(b+c-q)(b-c+q)}{2cq},\quad 1+\cos z=\frac{(c+q+b)(c+q-b)}{2cq},$$
$$1-\cos w=\frac{(c+b-q)(c-b+q)}{2bq},\quad 1+\cos w=\frac{(b+q+c)(b+q-c)}{2bq},$$

where $a = AB$, $b = BC$, $c = CD$, $d = DA$ and $q = BD$ in quadrilateral $ABCD$. Using these and the trigonometric identity

$$\tan^2 \frac{u}{2} = \frac{1 - \cos u}{1 + \cos u},$$

the second equality in the theorem is equivalent to

$$\frac{(d+a-q)^2(d-a+q)(a-d+q)(c+q+b)^2(c+q-b)(b+q-c)}{16abcdq^4}$$
$$= \frac{(b+c-q)^2(b-c+q)(c-b+q)(a+q+d)^2(a+q-d)(d+q-a)}{16abcdq^4}.$$

This is factored as

$$4qQ_1(a+d-b-c)\left((a+d)(b+c)-q^2\right) = 0 \tag{10}$$

where

$$Q_1 = \frac{(a-d+q)(d-a+q)(b-c+q)(c-b+q)}{16abcdq^4}$$

is a positive expression according to the triangle inequality. We also have that $a + d > q$ and $b + c > q$, so $(a + d)(b + c) > q^2$. Hence we have proved that

$$\tan \frac{x}{2} \tan \frac{y}{2} = \tan \frac{z}{2} \tan \frac{w}{2} \quad \Leftrightarrow \quad a + d = b + c$$

which according to (2) shows that the second equality in the theorem is a necessary and sufficient condition for an excircle outside of B or D.

The same kind of reasoning for the first equality in the theorem yields

$$4qQ_2(a+b-c-d)\left((a+b)(c+d)-q^2\right) = 0 \tag{11}$$

where $(a + b)(c + d) > q^2$ and

$$Q_2 = \frac{(a-b+q)(b-a+q)(d-c+q)(c-d+q)}{16abcdq^4} > 0.$$

Hence

$$\tan \frac{x}{2} \tan \frac{w}{2} = \tan \frac{y}{2} \tan \frac{z}{2} \quad \Leftrightarrow \quad a + b = c + d$$

which according to (1) shows that the first equality in the theorem is a necessary and sufficient condition for an excircle outside of A or C. □

6. Characterizations with escribed circles

All convex quadrilaterals $ABCD$ have four circles, each of which is tangent to one side and the extensions of the two adjacent sides. In a triangle they are called the excircles, but for quadrilaterals we have reserved that name for a circle tangent to the extensions of all four sides. For this reason we will call a circle tangent to one side of a quadrilateral and the extensions of the two adjacent sides an *escribed circle*.[8] The four of them have the interesting property that their centers form a cyclic quadrilateral. If $ABCD$ has an incircle, then its center is also the intersection of the diagonals in that cyclic quadrilateral [4, pp.1–2, 5].

[8] In triangle geometry the two names excircle and escribed circle are synonyms.

First we will prove a new characterization for when a convex quadrilateral has an incircle that concerns the escribed circles.

Theorem 5. *A convex quadrilateral with consecutive escribed circles of radii R_a, R_b, R_c and R_d is tangential if and only if*

$$R_aR_c = R_bR_d.$$

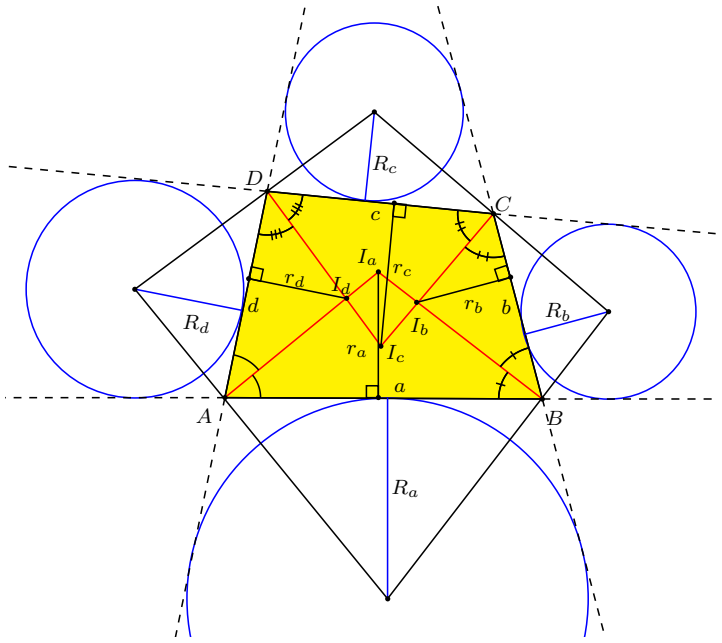

Figure 6. The four escribed circles

Proof. We consider a convex quadrilateral $ABCD$ where the angle bisectors intersect at I_a, I_b, I_c and I_d. Let the distances from these four intersections to the sides of the quadrilateral be r_a, r_b, r_c and r_d, see Figure 6. Then we have

$$\begin{aligned}
r_a\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right) &= a = R_a\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right),\\
r_b\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right) &= b = R_b\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right),\\
r_c\left(\cot\frac{C}{2}+\cot\frac{D}{2}\right) &= c = R_c\left(\tan\frac{C}{2}+\tan\frac{D}{2}\right),\\
r_d\left(\cot\frac{D}{2}+\cot\frac{A}{2}\right) &= d = R_d\left(\tan\frac{D}{2}+\tan\frac{A}{2}\right).
\end{aligned}$$

From two of these we get

$$r_b r_d \left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)\left(\cot\frac{A}{2}+\cot\frac{D}{2}\right) = R_b R_d \left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\left(\tan\frac{A}{2}+\tan\frac{D}{2}\right),$$

whence

$$r_b r_d \left(\frac{\cos\frac{A}{2}\sin\frac{D}{2}+\sin\frac{A}{2}\cos\frac{D}{2}}{\sin\frac{A}{2}\sin\frac{D}{2}}\right)\left(\frac{\cos\frac{B}{2}\sin\frac{C}{2}+\sin\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}\right) = R_b R_d \left(\frac{\sin\frac{B}{2}\cos\frac{C}{2}+\cos\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}\right)\left(\frac{\sin\frac{A}{2}\cos\frac{D}{2}+\cos\frac{A}{2}\sin\frac{D}{2}}{\cos\frac{A}{2}\cos\frac{D}{2}}\right).$$

This is equivalent to

$$\frac{r_b r_d}{R_b R_d} = \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\tan\frac{D}{2}. \tag{12}$$

By symmetry we also have

$$\frac{r_a r_c}{R_a R_c} = \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\tan\frac{D}{2}; \tag{13}$$

so

$$\frac{r_a r_c}{R_a R_c} = \frac{r_b r_d}{R_b R_d}. \tag{14}$$

The quadrilateral is tangential if and only if the angle bisectors are concurrent, which is equivalent to $I_a \equiv I_b \equiv I_c \equiv I_d$. This in turn is equivalent to that $r_a = r_b = r_c = r_d$. Hence by (14) the quadrilateral is tangential if and only if $R_a R_c = R_b R_d$. □

We also have the following formulas. They are not new, and can easily be derived in a different way using only similarity of triangles.

Corollary 6. *In a bicentric quadrilateral* [9] *and a tangential trapezoid with consecutive escribed circles of radii R_a, R_b, R_c and R_d, the incircle has the radius*

$$r = \sqrt{R_a R_c} = \sqrt{R_b R_d}.$$

Proof. In these quadrilaterals, $A + C = \pi = B + D$ or $A + D = \pi = B + C$ (if we assume that $AB \parallel DC$). Thus

$$\tan\frac{A}{2}\tan\frac{C}{2} = \tan\frac{B}{2}\tan\frac{D}{2} = 1$$

or

$$\tan\frac{A}{2}\tan\frac{D}{2} = \tan\frac{B}{2}\tan\frac{C}{2} = 1.$$

In either case the formulas for the inradius follows directly from (13) and (12), since $r = r_a = r_b = r_c = r_d$ when the quadrilateral has an incircle. □

[9] This is a quadrilateral that has both an incircle and a circumcircle.

In comparison to Theorem 5 we have the following characterizations for an extangential quadrilateral.

Theorem 7. *Let a convex quadrilateral $ABCD$ have consecutive escribed circles of radii R_a, R_b, R_c and R_d. The quadrilateral has an excircle outside one of the vertices A or C if and only if*

$$R_aR_b = R_cR_d$$

and an excircle outside one of the vertices B or D if and only if

$$R_aR_d = R_bR_c.$$

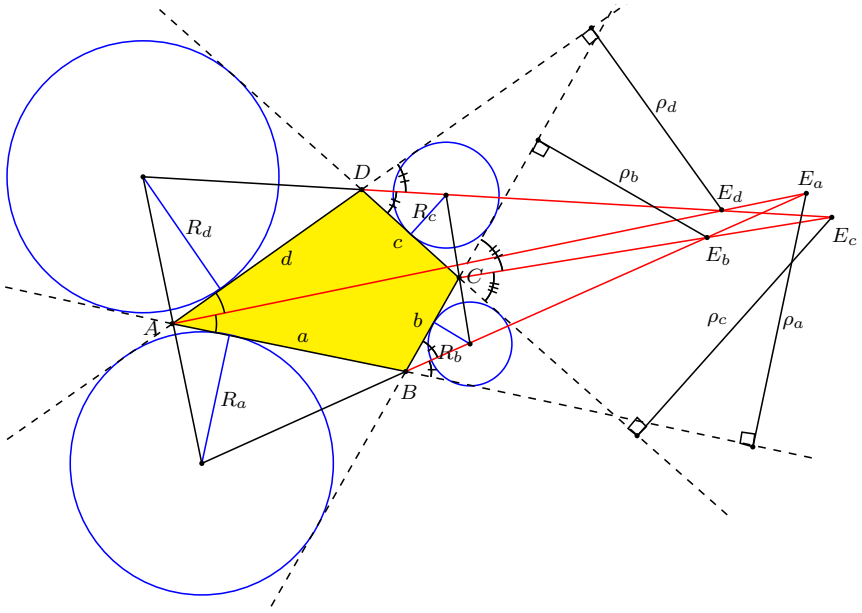

Figure 7. Intersections of four angle bisectors

Proof. We consider a convex quadrilateral $ABCD$ where two opposite internal and two opposite external angle bisectors intersect at E_a, E_c, E_b and E_d. Let the distances from these four intersections to the sides of the quadrilateral be ρ_a, ρ_c, ρ_b and ρ_d respectively, see Figure 7. Then we have

$$\rho_a\left(\cot\frac{A}{2}-\tan\frac{B}{2}\right)=a=R_a\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right),$$
$$\rho_b\left(\tan\frac{B}{2}-\cot\frac{C}{2}\right)=b=R_b\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right),$$
$$\rho_c\left(\tan\frac{D}{2}-\cot\frac{C}{2}\right)=c=R_c\left(\tan\frac{C}{2}+\tan\frac{D}{2}\right),$$
$$\rho_d\left(\cot\frac{A}{2}-\tan\frac{D}{2}\right)=d=R_d\left(\tan\frac{D}{2}+\tan\frac{A}{2}\right).$$

Using the first two of these, we get

$$\rho_a\rho_b\left(\cot\frac{A}{2}-\tan\frac{B}{2}\right)\left(\tan\frac{B}{2}-\cot\frac{C}{2}\right)$$
$$=R_aR_b\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right),$$

whence

$$\rho_a\rho_b\left(\frac{\cos\frac{A}{2}\cos\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}}{\sin\frac{A}{2}\cos\frac{B}{2}}\right)\left(\frac{\sin\frac{B}{2}\sin\frac{C}{2}-\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}\sin\frac{C}{2}}\right)$$
$$=R_aR_b\left(\frac{\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}\right)\left(\frac{\sin\frac{B}{2}\cos\frac{C}{2}+\cos\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}\right).$$

This is equivalent to

$$\rho_a\rho_b\frac{\cos\frac{A+B}{2}\left(-\cos\frac{B+C}{2}\right)}{\sin\frac{A}{2}\cos^2\frac{B}{2}\sin\frac{C}{2}}=R_aR_b\frac{\sin\frac{A+B}{2}\sin\frac{B+C}{2}}{\cos\frac{A}{2}\cos^2\frac{B}{2}\cos\frac{C}{2}},$$

which in turn is equivalent to

$$\frac{\rho_a\rho_b}{R_aR_b}=-\tan\frac{A+B}{2}\tan\frac{B+C}{2}\tan\frac{A}{2}\tan\frac{C}{2}. \tag{15}$$

By symmetry ($B\leftrightarrow D$), we also have

$$\frac{\rho_c\rho_d}{R_cR_d}=-\tan\frac{A+D}{2}\tan\frac{D+C}{2}\tan\frac{A}{2}\tan\frac{C}{2}. \tag{16}$$

Now using the sum of angles in a quadrilateral,

$$\tan\frac{A+B}{2}=-\tan\frac{D+C}{2}$$

and

$$\tan\frac{B+C}{2}=-\tan\frac{A+D}{2}.$$

Hence

$$\tan\frac{A+B}{2}\tan\frac{B+C}{2}=\tan\frac{A+D}{2}\tan\frac{D+C}{2}$$

so by (15) and (16) we have

$$\frac{\rho_a\rho_b}{R_aR_b}=\frac{\rho_c\rho_d}{R_cR_d}. \tag{17}$$

The quadrilateral is extangential if and only if the internal angle bisectors at A and C, and the external angle bisectors at B and D are concurrent, which is equivalent to $E_a\equiv E_b\equiv E_c\equiv E_d$. This in turn is equivalent to that $\rho_a=\rho_b=\rho_c=\rho_d$. Hence by (17) the quadrilateral is extangential if and only if $R_aR_b=R_cR_d$.

The second condition $R_aR_d=R_bR_c$ is proved in the same way. □

We have not found a way to express the exradius (the radius in the excircle) in terms of the escribed radii in comparison to Corollary 6. Instead we have the following formulas, which although they are simple, we cannot find a reference for. They resemble the well known formulas $r = \frac{K}{a+c} = \frac{K}{b+d}$ for the inradius in a tangential quadrilateral with sides a, b, c, d and area K.

Theorem 8. *An extangential quadrilateral with sides a, b, c and d has the exradius*

$$\rho = \frac{K}{|a-c|} = \frac{K}{|b-d|}$$

where K is the area of the quadrilateral.

Proof. We prove the formulas in the case that is shown i Figure 8. The area of the extangential quadrilateral $ABCD$ is equal to the areas of the triangles ABE and ADE subtracted by the areas of BCE and CDE. Thus

$$K = \tfrac{1}{2}a\rho + \tfrac{1}{2}d\rho - \tfrac{1}{2}b\rho - \tfrac{1}{2}c\rho = \tfrac{1}{2}\rho(a+d-b-c)$$

where the exradius ρ is the altitude in all four triangles. Hence

$$\rho = \frac{2K}{a-c+d-b} = \frac{K}{a-c} = \frac{K}{d-b}$$

since here we have $a+b = c+d$ (the excircle is outside of C), that is $a-c = d-b$. To cover all cases we put absolute values in the denominators. □

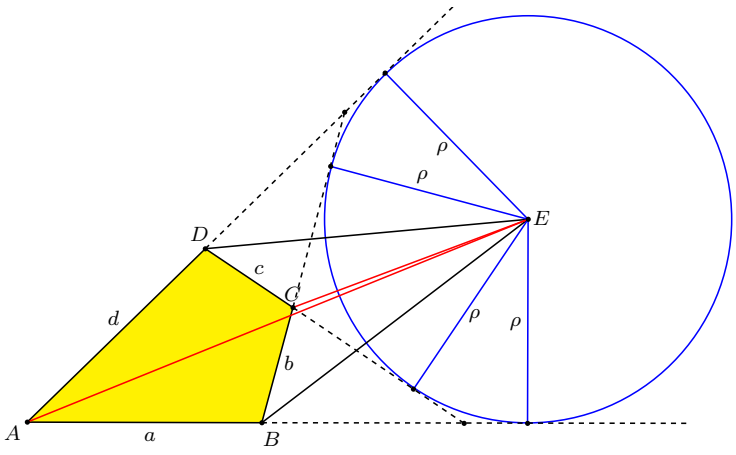

Figure 8. Calculating the area of $ABCD$ with four triangles

This theorem indicates that the exradii in all parallelograms (and hence also in all rhombi, rectangles and squares) are infinite, since in all of them $a = c$ and $b = d$.

References

[1] 48. Mathematik-Olympiade, 4. Stufe, Klasse 12-13 (in German), 2009, available at http://www.mathematik-olympiaden.de/aufgaben/48/4/A48134a.pdf

[2] A. Bogomolny, Inscriptible and exscriptible quadrilaterals, *Interactive Mathematics Miscellany and Puzzles*, http://www.cut-the-knot.org/Curriculum/Geometry/Pitot.shtml

[3] F. G.-M., *Exercices de Géométrie*, Cinquième édition (in French), Éditions Jaques Gabay, 1912.

[4] D. Grinberg, A tour around Quadrilateral Geometry, available at http://www.cip.ifi.lmu.de/~grinberg/TourQuadriPDF.zip

[5] H. Grossman, Urquhart's quadrilateral theorem, *The Mathematics Teacher*, 66 (1973) 643–644.

[6] M. Hajja, A very short and simple proof of "The most elementary theorem" of Euclidean geometry, *Forum Geom.*, 6 (2006) 167–169.

[7] L. Hoehn, A new formula concerning the diagonals and sides of a quadrilateral, *Forum Geom.*, 11 (2011) 211–212.

[8] M. Josefsson, More characterizations of tangential quadrilaterals, *Forum Geom.*, 11 (2011) 65–82.

[9] M. Josefsson, Characterizations of orthodiagonal quadrilaterals, *Forum Geom.*, 12 (2012) 13–25.

[10] K. S. Kedlaya, *Geometry Unbound*, 2006, available at http://math.mit.edu/~kedlaya/geometryunbound/

[11] N. Minculete, Characterizations of a tangential quadrilateral, *Forum Geom.*, 9 (2009) 113–118.

[12] D. Pedoe, The Most "Elementary Theorem" of Euclidean Geometry, *Math. Mag.*, 4 (1976) 40–42.

[13] M. Radić, Z. Kaliman and V. Kadum, A condition that a tangential quadrilateral is also a chordal one, *Mathematical Communications*, 12 (2007) 33–52.

[14] L. Sauvé, On circumscribable quadrilaterals, *Crux Math.*, 2 (1976) 63–67.

[15] D. Sokolowsky, A "No-circle" proof of Urquhart's theorem, *Crux Math.*, 2 (1976) 133–134.

[16] I. Vaynshtejn, N. Vasilyev and V. Senderov, Problem M1495, *Kvant* (in Russian) no. 6, 1995, pp. 27–28, available at http://kvant.mirror1.mccme.ru/djvu/1995_06.djvu

[17] W. C. Wu and P. Simeonov, Problem 10698, *Amer. Math. Monthly*, 105 (1998) 995; solution, ibid., 107 (2000) 657–658.

[18] A. Zaslavsky, Problem M1887, *Kvant* (in Russian) no. 3, 2004 p. 19, available at http://kvant.mirror1.mccme.ru/djvu/2004_03.djvu

Martin Josefsson: Västergatan 25d, 285 37 Markaryd, Sweden
E-mail address: martin.markaryd@hotmail.com